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Some Properties of a Five-Parameter Bivariate Probability Distribution

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PREFACE

The report contains the results of several papers related to modeling using a class of the bivariate gamma distribution. The separate papers contain loosely related subjects pertaining to this problem. Since the separate papers were prepared at different times during the contract period and have been submitted for publication in the open literature and each paper is intended to be self-contained, there is some redundancy in tables and illustrations.

Each of the papers in this report were extensions and/or generalizations of the results given in NASA TM-82483, entitled "A Bivariate Gamma Probability Distribution with Application to Gust Modeling," by O. E. Smith, S. I. Adelfang, and J. D. Tubbs. A modification of this paper is currently under review by Communications in Statistics.

The first paper in this report, entitled "A Note on the Ratio of Positively Correlated Gamma Variates," has been accepted for publication in Communications in Statistics and it presents some new analytical results using a class of the bivariate gamma distribution. Comparable results were available in the open literature using a different class of the bivariate gamma.

The second paper is entitled "A Method for Determining if Unequal Shape Parameters are Necessary in a Bivariate Gamma Distribution" and is an application of the results given in the first paper and addresses questions concerning

hypothesis tests for equality of shape parameters from correlated gamma distributed variates. This paper is currently under review by *Technometrics*.

The third paper, entitled "A Differential Equations Approach to the Modal Location for a Family of Bivariate Gamma Distribution," contains extensive analytical results for the location of the mode as a function of the free parameters. To the authors' knowledge this is the only such representation for a non-gaussian bivariate distribution. This paper has been submitted to *SIAM J. on Scientific and Statistical Computing*.

The fourth paper is a report summarizing the analysis of some wind gust data using the analytical results developed in relationship to the modeling application.

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CHAPTER I

A NOTE ON THE RATIO OF POSITIVELY CORRELATED GAMMA VARIATES

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ABSTRACT

Mielke and Flueck (1976) derived the density function and corresponding moments for the ratio of correlated gamma distributed variates. They considered a class of bivariate gamma distributions suggested by Cherian (1941) and David and Fix (1961). Recently, Lee, Holland, and Flueck (1979) derived some additional distributional results using this class of functions. This paper derives similar results using a different class of bivariate gamma distributions.

1. INTRODUCTION

Mielke and Flueck (1976) derived the distributional results for the ratio, R, of correlated gamma distributed variables. There are several classes of the bivariate gamma distribution [three are summarized in Mardia (1970) and an additional two in Johnson and

Kotz (1972)]. Mielke and Flueck (1976) derived the distributional results for the ratio, R, of correlated gamma distributed variables using the Cherian-David-Fix class of bivariate gamma random variables [Cherian (1941) and David and Fix (1961)]. That is, let X, Y, and P denote independent gamma random variables with common scale parameter λ and respective shape parameters $\alpha - \xi$, $\beta - \xi$, and ξ, for $0 < \xi < \min(\alpha,\beta)$.

Then it can be shown that the bivariate probability density function for $U = X + P$ and $V = Y + P$ is given by

$$f_{U,V}(u,v) = \frac{\exp[-(u+v)]}{K} \int_0^{\min(u,v)} p^{\xi-1}(u-p)^{\alpha-\xi-1}(v-p)^{\beta-\xi-1}e^{p}dp \qquad (1.1)$$

for $K = \Gamma(\alpha-\xi)\Gamma(\beta-\xi)\Gamma(\xi)$

when the scale parameter λ is assumed to be unity. Mielke and Flueck (1976) showed that (1.1) can be written as

$$f_{U,V}(u,v) = \begin{cases} \dfrac{u^{\alpha-1}v^{\beta-\xi-1}e^{-(u+v)}}{\Gamma(\alpha)\Gamma(\beta-\xi)} F_1^*(\xi,1+\xi-\alpha:\beta,u/v,-u) & \text{if } 0<u<v \\ \\ \dfrac{u^{\alpha-\xi-1}v^{\beta-1}e^{-(u+v)}}{\Gamma(\alpha-\xi)\Gamma(\beta)} F_1^*(\xi,1+\xi-\alpha:\beta,v/u,-v) & \text{if } 0<v<u \end{cases} \qquad (1.2)$$

where $F_1^*(a,b,c:x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_{n+m}(b)_n}{(c)_{m+n}m!n!} x^m y^n$ is a "degenerate" two variable hypergeometric function [Gradshteyn and Ryzhik (1967), p. 1067] and $(a)_n = \Gamma(a+n)/\Gamma(a)$. Thus, U and V are gamma random variables with shape parameters α and β and positive dependence parameter ξ. In particular, $E(U) = Var(U) = \alpha$, $E(V) = Var(V) = \beta$, and $Cov(U,V) = \xi$.

Mielke and Flueck (1976) derived the density function for $R = U/V$ using a change of variables. That is,

$$
f_R(r) = \begin{cases} \dfrac{r^{\alpha-1}(1+r)^{\xi-\alpha-\beta}}{B(\alpha-\xi,\beta)} \; F_1(\xi,\alpha+\beta-\xi,1+\xi-\beta,\alpha:r/1+r,r) & \text{if } 0 < r < 1 \\[2ex] \dfrac{r^{\alpha-\xi-1}(1+r)^{\xi-\alpha-\beta}}{B(\alpha-\xi,\beta)} \; F_1(\xi,\alpha+\beta-\xi,\ 1+\xi-\alpha,\beta:1/1+r,1/r) & \text{if } 1 < r \end{cases} \tag{1.3}
$$

where $F_1(a,b,c,d:x,y) = \sum_{m,n=0}^{\infty} \dfrac{(a)_{m+n}(b)_m(c)_n}{(d)_{m+n}m!n!} \; x^m y^n, \; |x| < 1 \; |y| < 1$

is a two variable hypergeometric function [Gradshteyn and Ryzhik, (1967), p. 1053], and $B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$. In addition, they show that the integral moments of R are given by

$$
E(R^s) = \sum_{j=0}^{s} \binom{s}{j} \frac{(\alpha-\xi)_j(\xi)_{s-j}}{(\beta-j)_s} \quad \text{for } s \geq 0. \tag{1.4}
$$

In particular,

$$
\begin{aligned} E(R) &= \alpha\beta-\xi/\beta(\beta-1), \quad \beta > 1 \\ E(R^2) &= \frac{\xi(\xi+1)}{\beta(\beta+1)} + \frac{2(\alpha-\xi)\xi}{\beta(\beta-1)} + \frac{(\alpha-\xi+1)(\alpha-\xi)}{(\beta-1)(\beta-2)}, \; \beta > 2 \end{aligned} \tag{1.5}
$$

Recently, Lee, Holland, and Flueck (1979) were able to obtain comparable results for density of R using the Cherian-David-Fix class of densities by expressing f_R as a weighted difference of hypergeometric functions. The purpose of this paper is to derive comparable results for R using a different class of the bivariate gamma distribution. This class is a special case of the one suggested by Jensen (1970) as modified by Gunst and Webster (1973). The next section contains a brief discussion of this class of distributions. In section 3 the derivation of f_R is given using this class of functions. Section 4 outlines a possible application for the probability function in the area of hypothesis testing for the equality of shape parameters in the presence of correlation.

2. GUNST AND WEBER CLASS OF BIVARIATE GAMMAS

Gunst and Weber (1973) proposed a computationally feasible method for deriving the joint density function for the bivariate chi-square distribution. Since the chi-square is a special case of the gamma, this method was used for the bivariate gamma case. That is, a bivariate gamma density function for U and V with common scale parameter $\lambda = 1$ and shape parameters α,β, $(\alpha < \beta)$ is given by

$$f(u,v) = \frac{u^{\alpha-1}v^{\beta-1}e^{-[(u+v)/(1-\eta)]}}{(1-\eta)^{\alpha}\Gamma(\alpha)\Gamma(\beta-\alpha)} \sum_{j}^{\infty}\sum_{k}^{\infty} \frac{\eta^{j+k}\Gamma(\beta-\alpha+k)(uv)^{j}v^{k}}{(1-\eta)^{2j+k}\Gamma(\beta+j+k)j!k!} \qquad (2.1)$$

where $\eta = \rho\sqrt{(\beta/\alpha)}$, ρ is the correlation coefficient between the variables U and V. Gunst and Webster (1973) suggested this class of densities in that they are computationally tractable and do not involve mathematical functions, such as Laguerre polynomials or convoluted sums [Jensen (1970) and Kibble (1941)]. Smith, Adelfang, and Tubbs (1982) discuss this class of densities in greater detail.

In the next section the distributional properties for the ratio, R, are derived using the Gunst-Webster class of bivariate gammas.

3. RATIO OF CORRELATED GAMMA VARIATES

By letting $R = U/V$ and $S = U+V$, the joint pdf for R and S can easily be shown to be

$$f_{R,S}(r,s) = c_1 \sum_{j}^{\infty}\sum_{k}^{\infty} c_2 \frac{s}{(1+r)^2}\left[\frac{sr}{1+r}\right]^{\alpha+j-1}\left[\frac{s}{1+r}\right]^{\beta+j+k-1} e^{-s/(1-\eta)} \qquad (3.1)$$

where $c_1 = [(1-\eta)^{\alpha}\Gamma(\alpha)(\beta-\alpha)]^{-1}$, $c_2 = \dfrac{\eta^{j+k}\Gamma(\beta-\alpha+k)}{(1-\eta)^{2j+k}\Gamma(\beta+j+k)j!k!}$. Hence, by integrating over S the pdf for R becomes

$$f_R(r) = (1-\eta)^\beta \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} c_j c_k \, B(\alpha+j,\beta+j+k)^{-1} r^{\alpha+j-1}/(1+r)^{\alpha+\beta+2j+k} \quad (3.2)$$

where $c_j = (\alpha)_j \eta^j/j!$, $c_k = (\beta-\alpha)_k \eta^k/k!$, $(a)_n = \Gamma(a+n)/\Gamma(a)$, and $B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$.

Whenever the shape parameters are equal then the density function for R is given by

$$f_R(r) = (1-\eta)^\alpha \sum_{j=0}^{\infty} c_j \, B(\alpha+j,\alpha+j)^{-1} r^{\alpha+j-1}/(1+r)^{2\alpha+2j} \quad (3.3)$$

From (3.2) it can be shown that the m^{th} raw moment for R is given by

$$E(R^m) = (1-\eta)^\beta \sum_{j=0}^{\infty} c_j \sum_{k=0}^{\infty} c_k \, B(\alpha+j+m,\beta+j+k-m)/B(\alpha+j,\beta+j+k) \quad (3.4)$$

if $m < \beta$. In which case, it follows that

$$E(R) = (1-\eta)^\beta \sum_{j}^{\infty} c_j \sum_{k}^{\infty} c_k \, (\alpha+j)/(\beta+j+k-1) \quad (3.5)$$

and

$$E(R^2) = (1-\eta)^\beta \sum_{j}^{\infty} c_j \sum_{k}^{\infty} c_k \, (\alpha+j)(\alpha+j+1)/(\beta+j+k-1)(\beta+j+k-2) \quad (3.6)$$

Whenever $\eta = 0$, then

$$E(R) = \alpha/(\beta-1), \; E(R^2) = \alpha(\alpha+1)/(\beta-1)(\beta-2) \quad (3.7)$$

which agrees with the values given by Mielke and Flueck (1976) whenever $\xi = 0$ and with Lee, Holland, and Flueck (1979) whenever $a = 0$.

Lee, Holland, and Flueck (1979) discuss some of the mathematical properties for the density of R for various values of

α,β, and η. They demonstrated that the density can be ∞ at $r=1$ whenever either of the shape parameters is less than one. However, in the Gunst-Webster construction by assuming that $\alpha > 1$ and $\alpha < \beta$ the density function given in equation (3.2) is stable. Figures 1-4 illustrate the various shapes that $f_R(r)$ has as a function of the three parameters.

A definite computational advantage of equation (3.2) versus equation (1.3) stems from the ability to compute the tail probabilities for R. By letting $a=\alpha+j$ and $b=\beta+j+k$, we have

$$F_R(r_0) = (1-\eta)^{\beta} \sum_j^{\infty} c_j \sum_k^{\infty} c_k \, P[F_{2a,2b} \leq br_0/a] \tag{3.8}$$

where $F_{r,s}$ denotes a random variable from an F-distribution with r and s degrees of freedom. Note if $\eta = 0$, then (3.7) becomes

$$F_R(r_0) = P[F_{2\alpha,2\beta} \leq \beta r_0/\alpha] \tag{3.9}$$

which agrees with the well known results concerning the ratio of independent chi-squares. Furthermore, if $\eta \neq 0$ and $\alpha = \beta$ then (3.7) becomes

$$F_R(r_0) = (1-\eta)^{\alpha} \sum_j^{\infty} c_j \, P[F_{2(\alpha+j),2(\alpha+j)} \leq r_0] \tag{3.10}$$

which is similar to an expression given by Johnson and Kotz (1972), Chapter 40, Section 3.

4. APPLICATION

In this section an application is given for computing the cdf of R, given by equation (3.7). Diagram 1 defines the area given in equation (4.1).

FIGURE 1. DENSITY FUNCTION FOR R = U/V α = 1.5, β = 1.5

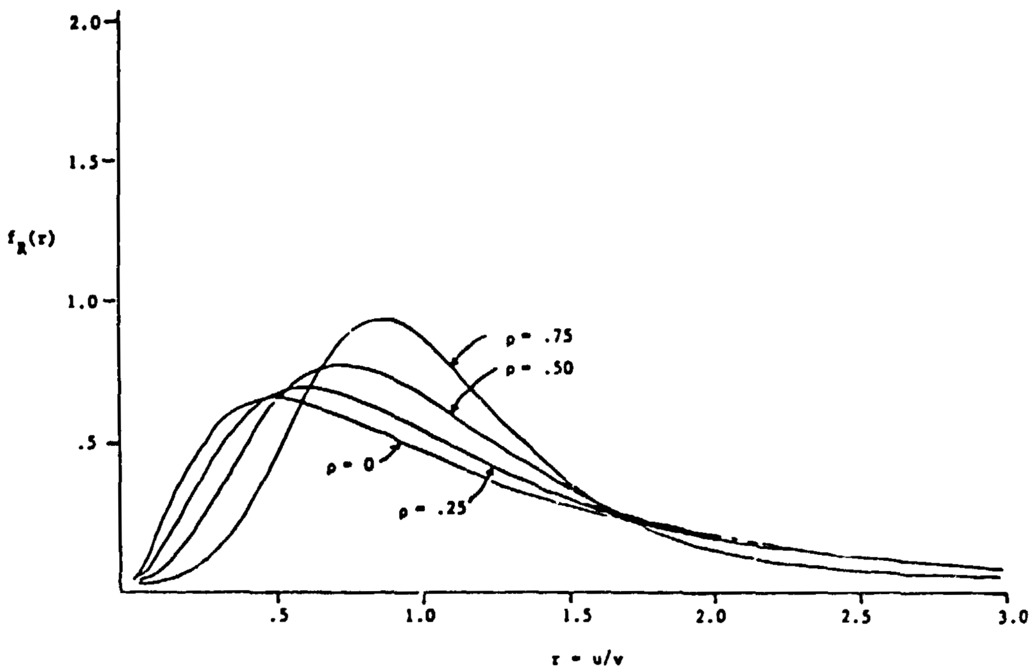

FIGURE 2. DENSITY FUNCTION FOR R = U/V α = 1.0, β = 3.0

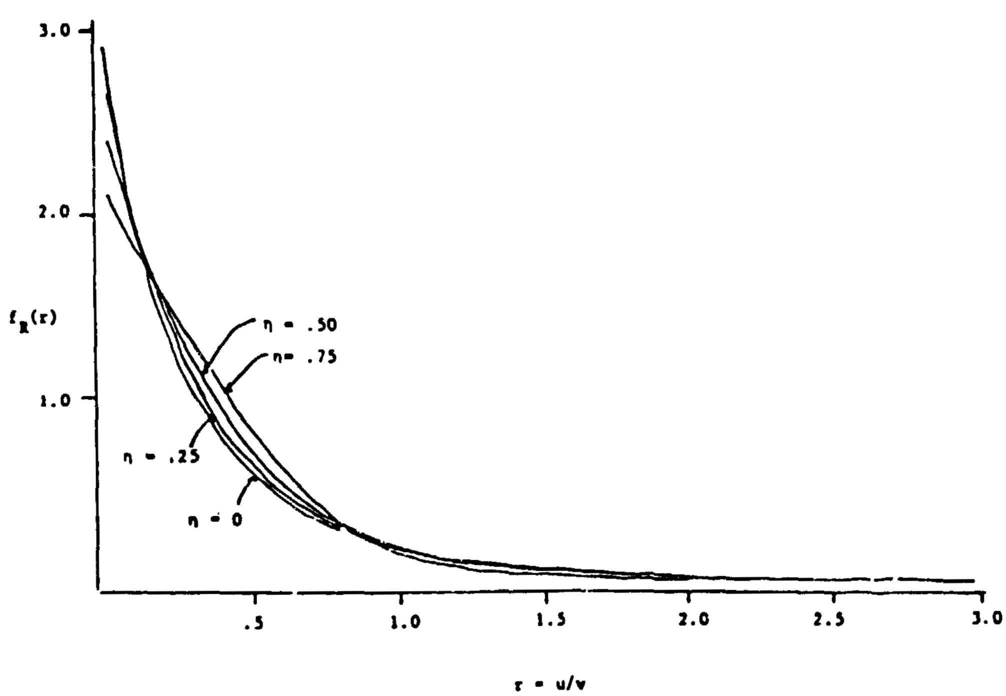

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FIGURE 3. DENSITY FUNCTION FOR R = U/V α = 2.0, β = 3.0

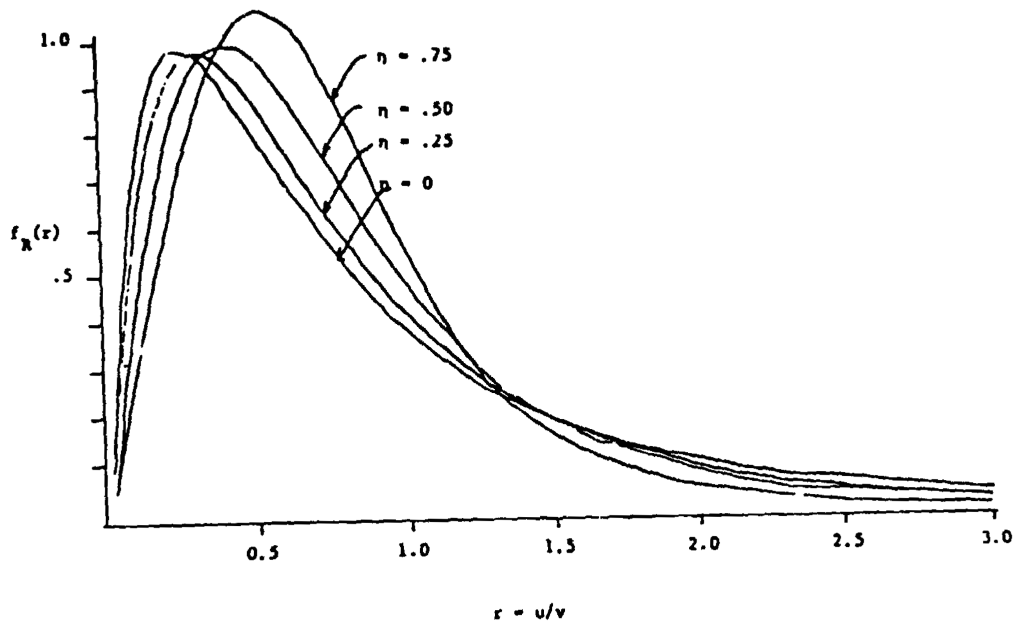
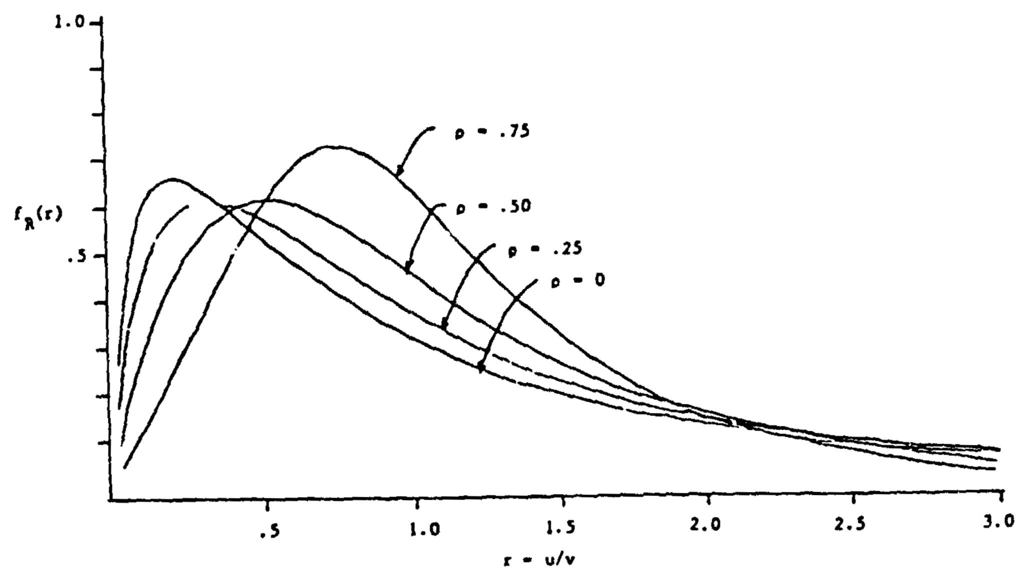


FIGURE 4. DENSITY FUNCTION FOR R = U/V α = 3.0, β = 3.0



By letting $\cot \theta_0 = U/V = r_0$ and $G(\theta) = 1-F_R(r_0)$, one has

$$G(\theta) = (1-\eta)^\beta \sum_j^\infty c_j \sum_k^\infty c_k P[F_{2b,2a} \leq (a/b)\tan\theta] \tag{4.1}$$

Figure 5 contains the graph of the function $G(\theta)$ versus θ for $\alpha = 1$ and $\beta = 1, 2,$ or 3 and $\eta = 0, .25, .50,$ and $.75$. From this figure and other cases which are not included one observes that whenever $\alpha = \beta$ then $G(45^\circ) = .5$ and $G(45^\circ) < .5$ whenever $\alpha < \beta$. This observation and additional properties were used in developing a test for the hypothesis

$$H_0: \alpha = \beta \quad \text{vs.} \quad H_A: \alpha < \beta \tag{4.2}$$

The procedure is presented in Tubbs (1983) and uses the Cramer-Von Mises criteria for testing (4.2). That is, define

$$W_n = n \int \{F_R(r) - F_n(r)\}^2 dF_R(r) \tag{4.3}$$

where $F_R(r)$ is the cdf for the null distribution given in (3.10). $F_n(r)$ is the empirical distribution for $r_i = u_i/v_i$ and the r_i's are arranged in increasing order. Whenever H_0 is true, then W_n is distribution free and has a convenient computational form given by

$$W_n = \frac{1}{12n} + \sum_{i=1}^n \{Z_i - \frac{(2i-1)}{2n}\}^2 \tag{4.4}$$

where $Z_i = F_R(r_i)$. H_0 is rejected if W_n exceeds a specified critical point. Tubbs (1983) considers the properties of this test procedure in greater detail.

5. CONCLUSIONS AND SUMMARY

This paper derives both the density and the distribution functions for the ratio of positively correlated gamma variates using a modification of Jensen's bivariate gamma distribution. The expression for the moments differ from those given by either Mielke and Flueck (1976) or Lee, Holland, and Flueck (1979).

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However, all the expressions are identical whenever the variates are uncorrelated. A principal advantage found in this representation stems from the ability to compute the CDF of the ratio. The value of the CDF for the ratio was shown to have potential application to the problem of testing for equality of shape parameters in a particular family of the bivariate gamma distribution.

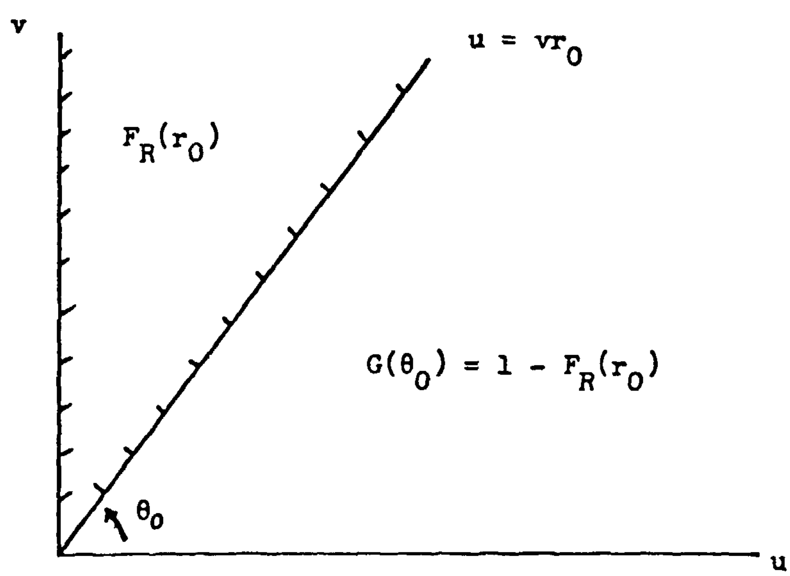

Diagram 1

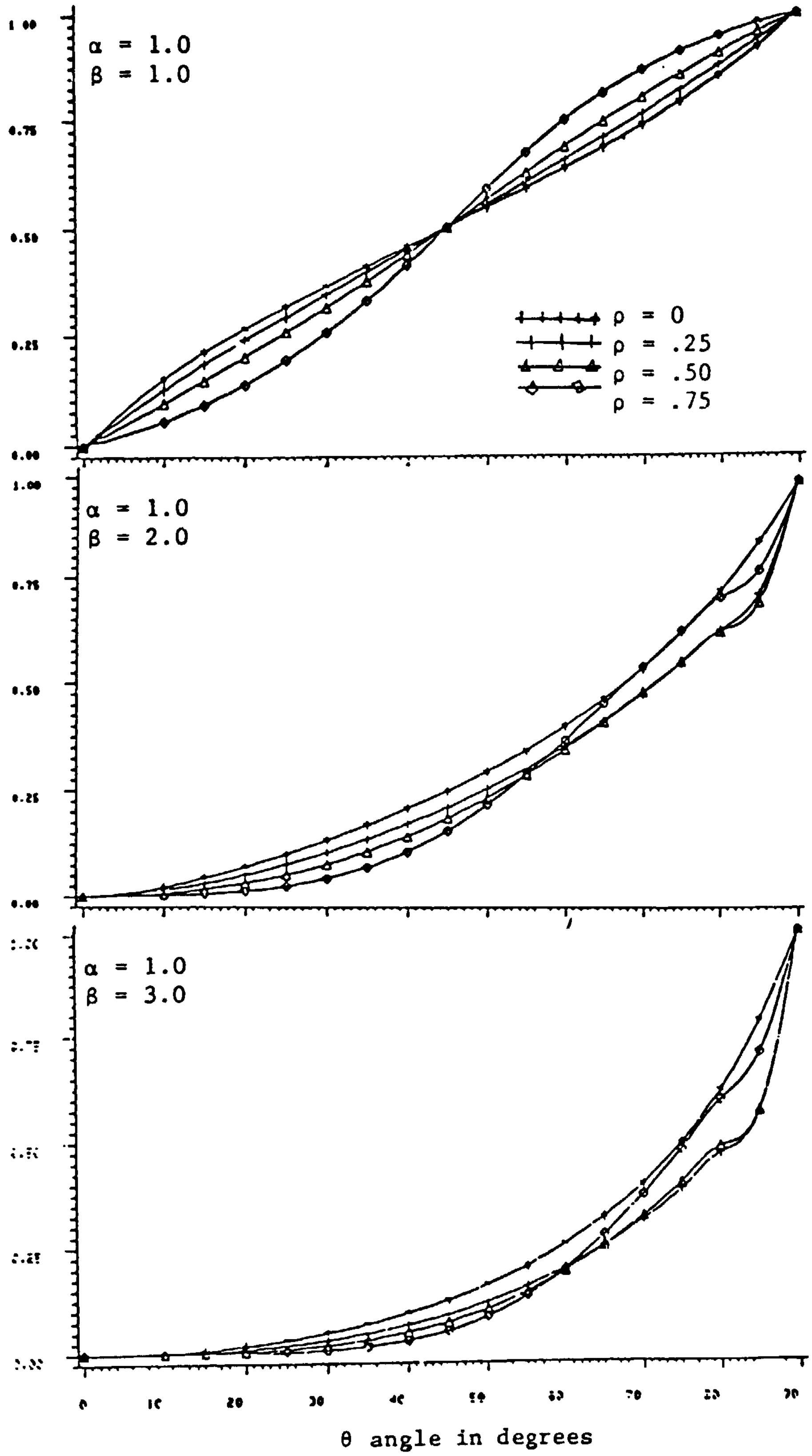

FIGURE 5. G(θ) vs. θ

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CHAPTER II

A METHOD FOR DETERMINING IF UNEQUAL SHAPE PARAMETERS ARE NECESSARY IN A BIVARIATE GAMMA DISTRIBUTION

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ABSTRACT

A procedure for aiding an experimentalist in deciding between four and five parameters in a Jensen's type bivariate gamma distribution is presented. The procedure is based upon the properties of the CDF for the ratio of correlated gamma distributed variates. The criteria of interest is posed in a test of hypothesis setting and results are presented using the Cramér-Von Mises test of fit.

1. INTRODUCTION

Smith and Adelfang (1981) discuss the applicability of a bivariate gamma distribution as a parametric model for wind gust amplitude and length. In modeling this bivariate data with a gamma distribution, it was necessary to find a distribution that would allow for correlation between the random variables X and Y when the marginal distributions are univariate gammas with possibly unequal shape and scale parameters. That is, $X \sim G(\gamma_x, \beta_x)$ and

$Y \sim G(\gamma_y, \beta_y)$ where the probability density function for $Z \sim G(\gamma, \beta)$ is given by

$$f_Z(z) = \beta^\gamma z^{\gamma-1} e^{-\beta z}/\Gamma(\gamma). \tag{1.1}$$

A brief survey of the open literature reveals that there are several classes of the bivariate gamma distribution. One need only consult Mardia (1970) and Johnson and Kotz (1972) to find five classes of the bivariate gamma distribution [Kibble (1941), Cherion (1941), McKay (1934), Jensen (1970), and Moran (1969)]. Of these classes only Jensen (1970) and Moran (1969) allow for unequal shape parameters and both of these have computational limitations which affect their utility to the experimentalist. Recently, McAllister, Lee, and Holland (1981) and McAllister (1983) have addressed the limitations with Jensen's model and provided results which overcome many of the computational difficulties. However, at the time of Smith et al. (1983) development these results were not available. Hence, they modified a bivariate chi-square model given by Gunst and Webster (1973). This allows for possibly unequal shape parameters and is computationally tractable. The model is not as general as that given by Jensen (1970), however, one can derive the bivariate model given by Kibble (1941) as a special case whenever the shape parameters are equal. In this paper, the unequal shape parameter model will be referred to as the five-parameter model and the equal shape

case as the four-parameter model. Smith, Adelfang, and Tubbs (1983) discuss the properties of these distributions and it is apparent that the four-parameter has numerous computational advantages over the five-parameter model. So if one assumes that the data is correctly modeled by this class of the bivariate gamma distribution, a question of practical interest becomes, How does one decide if the five-parameters are really necessary? The purpose of this paper is to present a procedure which would aid the experimentalist in answering the above question. The problem is posed in a hypothesis testing setting. That is, test the hypothesis

$$H_0: \gamma_x = \gamma_y \tag{1.2}$$

versus

$$H_1: \gamma_x < \gamma_y. \tag{1.3}$$

It should be noted that the proposed method is not an omnibus test of fit for the bivariate gamma against all other possible models. Instead the procedure is intended for deciding between the four or five parameter models as given in Smith, Adelfang, and Tubbs (1983).

The next section contains the distributional results needed for the test of hypothesis (1.2). The test procedure is given in section 3 and evaluated in section 4. Section 5 contains a summary and remarks concerning some of the limitations of the procedure.

2. DISTRIBUTIONAL RESULTS

Smith, Adelfang, and Tubbs (1983) modified a bivariate-Chi square distribution given by Gunst and Webster (1973) and obtained the density function given by

$$f(x,y) = K_1/K_2 \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} c_{jk} x^j (\eta y)^{j+k} \tag{2.1}$$

where

$$K_1 = x^{\gamma_x - 1} y^{\gamma_y - 1} \exp -\{(x+y)/(1-\eta)\},$$

$$K_2 = (1-\eta)^{\gamma_x} \Gamma(\gamma_x) \Gamma(\gamma_y - \gamma_x),$$

$$c_{jk} = \eta^{j+k} \Gamma(\gamma_y - \gamma_x + k)/\{(1-\eta)^{2j+k} \Gamma(\gamma_y + j + k)\ j!\ k!\},$$

and $x = X\beta_x$, $y = Y\beta_y$, β_x, β_y are known scale parameters, $\eta = \rho \sqrt{\gamma_y/\gamma_x}$, ρ is the correlation coefficient between the variables X and Y. The joint probability distribution function is given by

$$\begin{aligned} F(x_o,y_o) &= \Pr[X \leq x_o,\ Y \leq y_o] \\ &= J \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} d_{jk}\ H(\gamma_x + j, x_o/(1-\eta)) \\ &\quad \cdot H(\gamma_y + j + k,\ y_o/(1-\eta)) \end{aligned} \tag{2.2}$$

where

$$J = (1-\eta)^{\gamma_y}/\Gamma(\gamma_x)\Gamma(\gamma_y - \gamma_x),$$

$$d_{jk} = \eta^{j+k} \Gamma(\gamma_y - \gamma_x + k)/\Gamma(\gamma_y + j + k)\ j!\ k!$$

$$H(a,x) = \int_0^{\infty} t^{a-1}\ e^{-t}\ dt.$$

Equations (2.1) and (2.2) are for the unequal shape parameters and will be referred to as the five-parameter model. It should be re-emphasized that this model is not completely general in that one assumes that $\gamma_y > \gamma_x$ and the correlation between variables X and Y are restricted to the interval $[0, \eta\sqrt{\gamma_x/\gamma_y}]$ for $\eta\varepsilon[0,1]$.

If $\gamma_x = \gamma_y = \gamma$ then it can be shown that (2.1) and (2.2) reduce to the well known functions given by Kibble (1941). That is, the density function is given by

$$f(x,y) = (xy)^{\gamma-1}\exp-\{(x+y)/(1-\eta)\}/\Gamma(\gamma) \cdot \sum_{j=0}^{\infty} (\eta xy)/(1-\eta)^2)^j/\Gamma(\gamma+j)\ j! \tag{2.3}$$

and the distribution function becomes

$$F(x,y) = (1-\eta)^{\gamma}/\Gamma(\gamma) \sum_{j=0}^{\infty} \eta^j/\Gamma(\gamma+j)\ j! \cdot H(\gamma+j, x/(1-\eta))\ H(\gamma+j, y/(1-\eta)). \tag{2.4}$$

Equations (2.3) and (2.4) will be referred to as the four parameter model. A comparison of the distribution function given in (2.2) and (2.4) reveals that there are distinct differences in terms of the computational complexity. Thus for computational reasons the experimentalist would like to know how much greater does $\hat{\gamma}_y$ have to exceed $\hat{\gamma}_x$ before equation (2.2) is really necessary. Ideally he would like to answer this question before using both (2.2) and (2.4) then selecting the results which are more gratifying. In order to address this issue, this

paper considers the problem of testing hypothesis (1.2) versus (1.3) using an univariate random variate given by the ratio of X to Y, $R = X/Y$. Tubbs and Smith (1983) derive the density and distribution functions for R whenever the bivariate density is either (2.1) or (2.3). That is, if equation (2.1) holds then the density function for R is given by

$$f_R(r) = (1-\eta)^{\gamma_y} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} c_j c_k B(a,b)^{-1} r^{a-1}/(1+r)^{a+b} \quad (2.5)$$

where $B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$, $c_j = (a)_j \eta^j/j!$,

$c_k = (b-a)_k \eta^k/k!$, $a = \gamma_x+j$, $b = \gamma_y+j+k$, and $(a)_n = \Gamma(a+n)/\Gamma(a)$.

The distribution function for R is given by

$$F_R(r_o) = \Pr[X/Y \leq r_o]$$

$$= (1-\eta)^{y} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} c_j c_k \Pr[F_{2a,2b} \leq br_o/a] \quad (2.6)$$

where $F_{r,s}$ denotes a random variable from an F-distribution with r and s degrees of freedom. The corresponding functions whenever $\gamma_x = \gamma_y = \gamma$ are given by

$$f_R(r) = (1-\eta)^{\gamma} \sum_{j=0} c_j \; B(a,a)^{-1} r^{a-1}/(1+r)^{2a} \quad (2.7)$$

and

$$F_R(r_o) = (1-\eta)^{\gamma} \sum_{j=0} c_j \; \Pr[F_{2a,2a} \leq r_o] \quad (2.8)$$

where $a = \gamma + j$.

3. HYPOTHESIS TESTING

Since $R = X/Y$ is a univariate random variable it is informative to graph $F_R(r)$ versus r. However, since $r > 0$ a more meaningful graph can be produced by letting $\theta = \cot^{-1} r$ and $G(\theta_o) = 1-F_r(r_o)$ where $\theta_o = \cot^{-1} r_o$. The area corresponding to $F_r(r_o)$ is shown in diagram 1. Furthermore, it follows that

$$G(\theta_o) = (1-\eta)^{\gamma_y} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} c_j c_k \Pr[F_{2a,2b} \leq (a/b) \tan\theta_o] \tag{3.1}$$

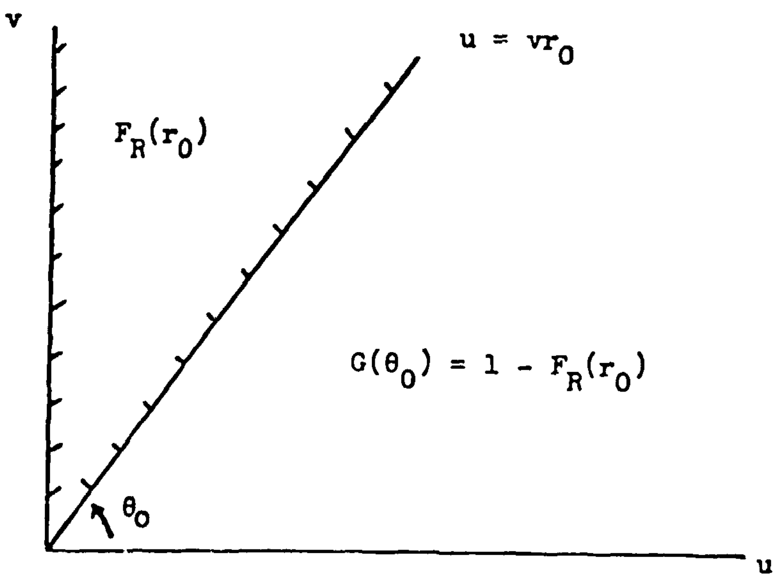
in the five-parameter model and

$$G(\theta_o) = (1-\eta)^{\gamma} \sum_{j=0}^{\infty} c_j \Pr[F_{2a,2a} \leq \tan\theta_o] \tag{3.2}$$

in the four-parameter case.

Since θ is restricted to the finite interval $(0, \pi/2)$, it is somewhat instructive to plot $G(\theta)$ versus θ as functions of the free parameters, γ_x, γ_y and η. As in Tubbs and Smith (1983) the scale parameters are assumed to be known and hence equal to one. This restriction will be addressed later in the paper. Figures 1-3 contain some of the illustrative cases.

From these plots one observes that $G(45^o) = .5$ whenever the four-parameter model holds and $G(45^o) < .5$ in the five-parameter models. Rather than just using this observation a function was selected to measure the distance between these distribution functions. The Cramér-Von Mises



v
$u = vr_0$
$F_R(r_0)$
$G(\theta_0) = 1 - F_R(r_0)$
θ_0
u

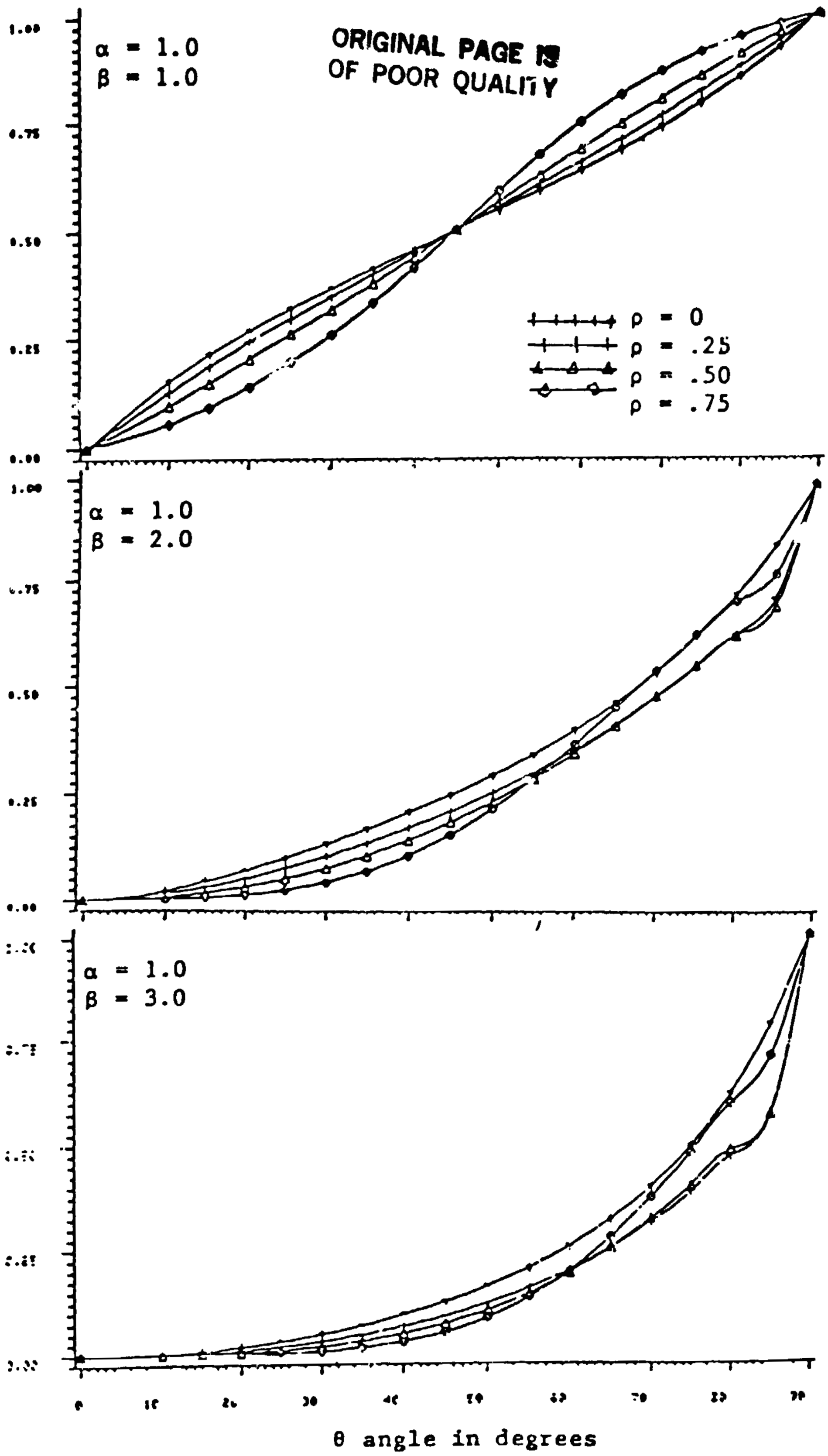


FIGURE 1. $F_R(\theta)$ vs. θ

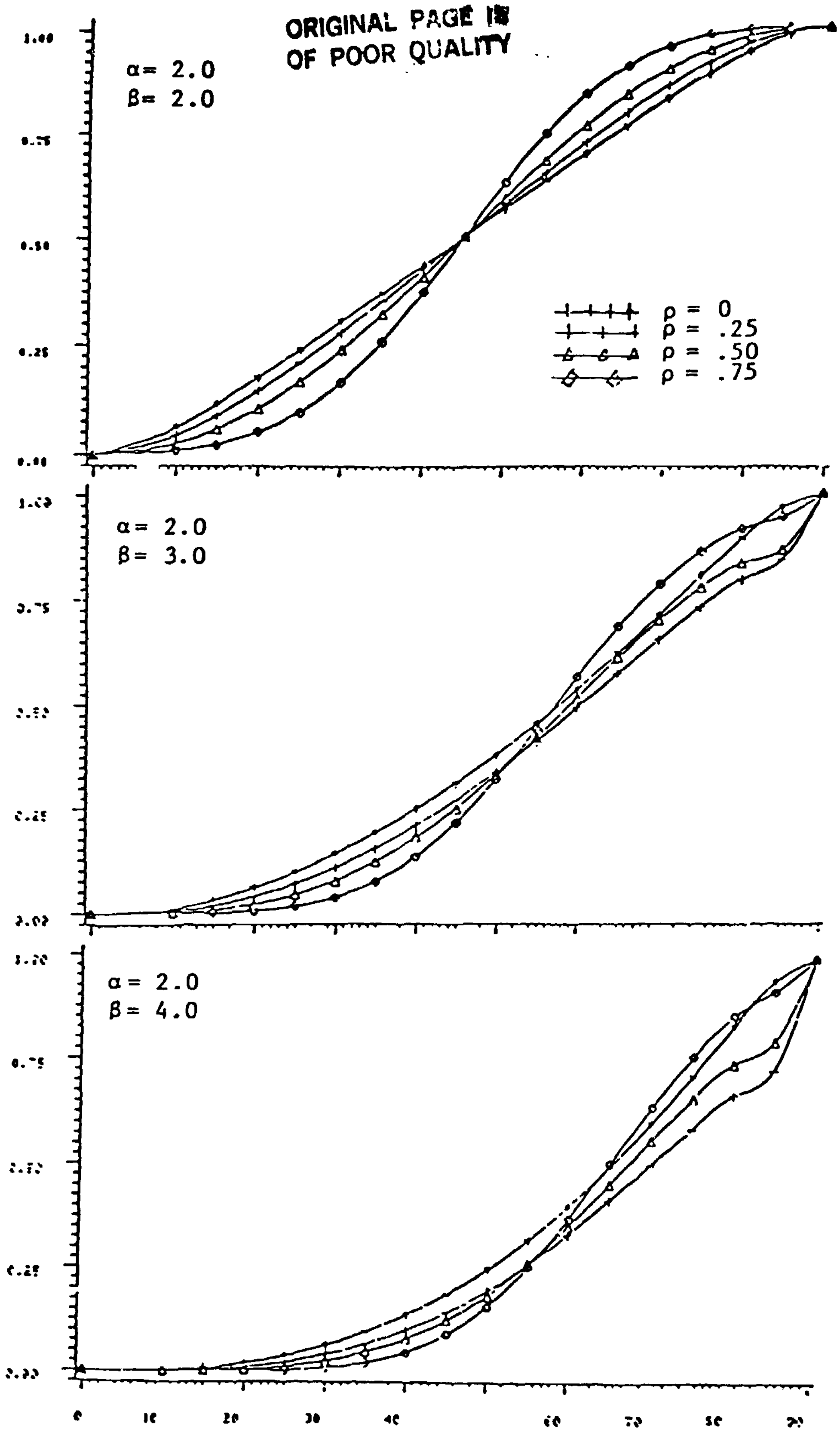


FIGURE 2. $F_R(\theta)$ vs. θ

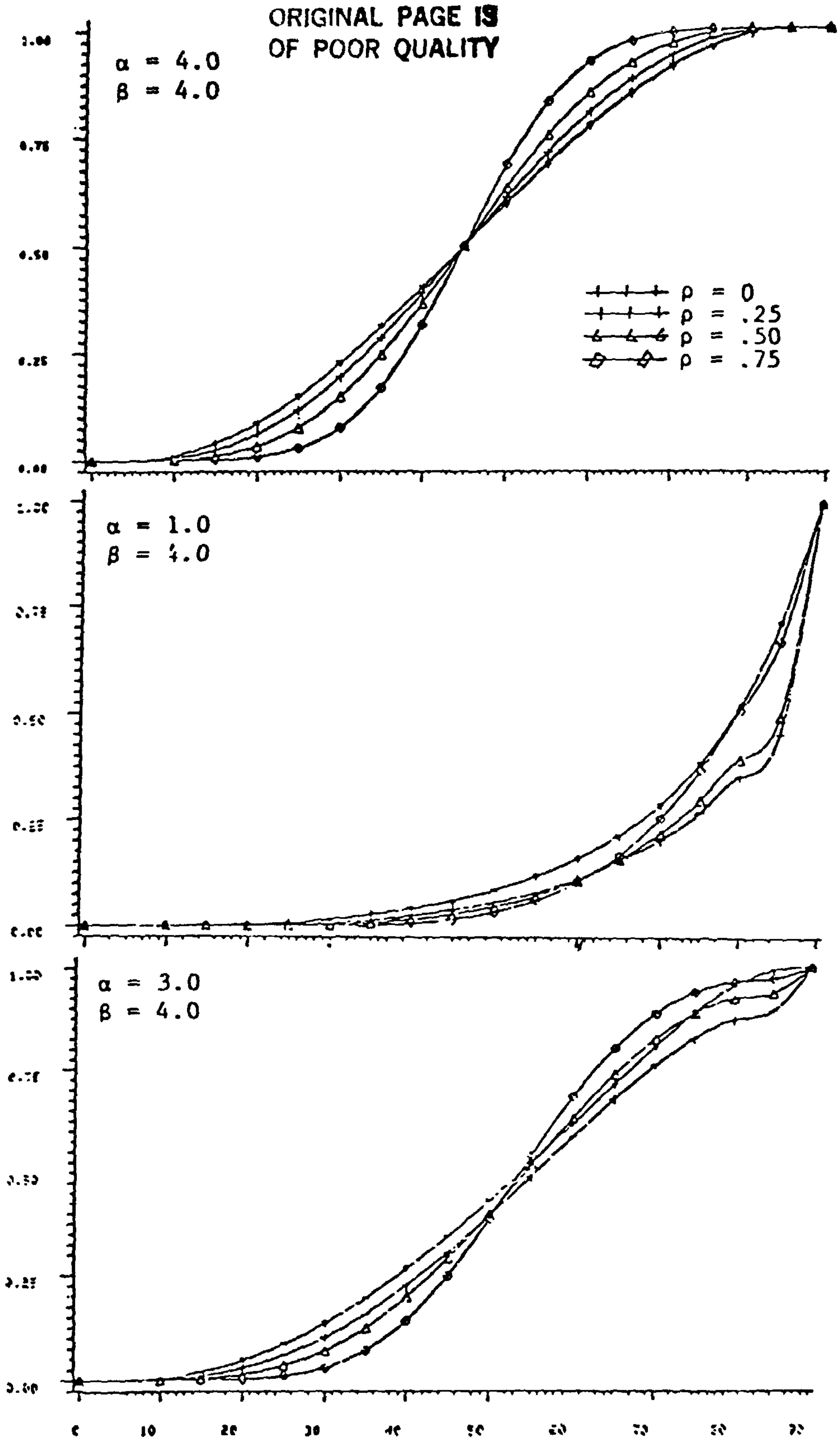


FIGURE 3. $F_R(\theta)$ vs. θ

type goodness-of-fit procedure was selected since the test is distribution free whenever the parameters are specified. Furthermore the test statistic is easy to compute.

Let

$$W_n = n\int_0^{\pi/2} \{G(\theta) - G_n(\theta)\}^2 dG(\theta) \tag{3.3}$$

where $G(\theta)$ is given in (3.2), $G_n(\theta)$ is the empirical distribution function of $\theta_i = \tan^{-1}(r_i)$, $r_i = X_i/Y_i$ are arranged in increasing order. Whenever hypothesis (1.2) is true, then W_n has the convenient computational form given by

$$W_n = 1/12n + \sum_{i=1}^{n} \{z_i - \frac{(2i-1)}{2n}\}^2 \tag{3.4}$$

where $z_i = G(\theta_i)$. Furthermore, from Anderson and Darling (1951) one can reject (1.2) whenever W_n exceeds a specified critical point. These critical points are given from Anderson and Darling's asymptotic distribution. Stephen (1976) defines a procedure for modifying the critical points for small samples, however, the underlying problem of modeling bivariate data will probably dictate large sample sizes.

4. EVALUATION OF THE TEST PROCEDURE

In this section the procedure defined in the previous section is evaluated. The evaluation is performed in two parts. The intent of the first part was to determine whether or not the procedure even works. That is, are the

apparent visual differences between the function $G(\theta)$ as seen in Figures 1-3 significant in the "Cramér-Von Mises" metric. The second part of the evaluation concerns the robustness of the procedure to the nuisance parameters.

In the first part, let

$$D_n(\delta) = n\int_0^{\pi/2} \{G(\theta) - A(\theta)\}^2 dG(\theta) \tag{4.1}$$

where $G(\theta)$ is given in (3.2) and $A(\theta)$ is given by (3.1) when $\gamma_y = \gamma_x + \delta$, for $\delta > 0$. For positive integers n, compute.

$$\alpha_n(\delta) = P_r[W_n > D_n(\delta)]. \tag{4.2}$$

If the alternative hypothesis given by

$$H_1: \gamma_x < \gamma_y = \gamma_x + \delta \tag{4.3}$$

holds, then the expected value of W_n in (3.3) is given by $D_n(\delta)$. Hence, $\alpha_n(\delta)$ is the expected type I error of testing hypothesis (1.2) as a function of δ. Table 1 contains the value of $\alpha_n(\delta)$ for various values of the parameters. The $\alpha_n(\delta)$'s were computed using Tiku's approximation to the asymptotic distribution of W_n [Tiku (1965)].

For example, from Table 1 one would expect the test to reject integer (δ=1) differences between the shapes for X and Y at the 95% significant level whenever n > 50.

The procedure used to generate the values in Table 1 is somewhat unconventional; however, they do indicate that

the test procedure would be sensitive to differences in the shape parameters that exceed unity. A Monte Carlo simulation was also performed. The results are not reported in the interest of space and since the simulation was quite limited. A detail simulation is very expensive due to the computational costs in computing the null distribution $G(\theta)$ needed in evaluating type I errors. It is especially costly to simulate any type II errors. In spite of these restrictions upon the simulation's merit, the results were supportive of the expected results given in Table 1.

The second part of the evaluation is concerned with the question of robustness of the test to the unspecified parameters, namely, ρ and β_x, β_y. In order to determine the sensitivity of the test to the misspecified correlation coefficient ρ, the following distance was evaluated for different values of $\gamma_x = \gamma_y$.

$$D_n(\rho) = n \, {}_{0}^{\pi/2}\{G(\theta) - B(\theta)\}^2 \, dG(\theta) \tag{4.4}$$

where $G(\theta)$ is given in equation (3.2) when $\rho = 0$, and $B(\theta)$ is given by equation (3.2) whenever $\rho > 0$, for $\rho = .25(.25).75$. Table 4 contains the type 1 errors $\alpha_n(\rho)$ given by

$$\Pr[W_n > D_n(\rho)] = \alpha_n(\rho) \tag{4.5}$$

for different values of n and $\gamma_x = \gamma_y = \gamma$. $\alpha_n(\rho)$ in

Table 1. Tail Probabilities for $\alpha_n(\delta)$*

γ_x	η	n	δ=.25	.50	.75	1.00	1.25	1.50	1.75	2.00
1	0	20	1.00	.45	.22	.12	.06	.04	.02	.02
		50	.53	.12	.02	.01				
		100	.25	.02	.01					
	.25	20	1.00	.41	.18	.09	.05	.03	.02	.01
		50	.48	.09	.02	.01				
		100	.21	.01						
	.50	20	.87	.33	.13	.96	.03	.02	.01	.01
		50	.41	.06	.01					
		100	.15	.01						
	.75	20	.69	.21	.07	.02	.01	.01		
		50	.27	.02						
		100	.06							
2	0	20	1.00	.82	.48	.27	.17	.13	.07	.04
		50	1.00	.36	.13	.04	.02			
		100	.60	.13	.02	.01	.01			
	.25	20	1.00	.72	.40	.22	.13	.07	.04	.03
		50	.91	.29	.09	.01				
		100	.52	.09	.01					
	.50	20	1.00	.59	.30	.15	.08	.04	.03	.01
		50	.73	.21	.05	.02				
		100	.41	.05	.01					
	.75	20	1.00	.40	.16	.06	.03	.01		
		50	.52	.09	.01					
		100	.24	.02						
3	0	20	1.00	1.00	.66	.43	.28	.18	.12	.08
		50	1.00	.55	.25	.11	.04	.02	.01	
		100	.87	.26	.07	.01				
	.25	20	1.00	1.00	.56	.35	.21	.13	.08	.05
		50	1.00	.46	.18	.07	.02	.01		
		100	.59	.11	.01					
	.75	20	1.00	.56	.26	.12	.05	.02	.01	
		50	.72	.18	.04	.01				
		100	.39	.04	.01					

*if $\alpha_n(\delta) < .01$, then the entry is left blank.

(4.5) is the expected type I error as a function of the nuisance parameter ρ. It should be mentioned that the distribution for W_n is not the same as that given by Anderson and Darling asymptotic approximation since the nuisance parameter ρ is unspecified [cp., Stephen (1976)], however, it does not appear feasible to follow Darling's procedure for computing the exact distribution whenever ρ and β_x, β_y are replaced by their consistent estimators. In spite of this shortcoming, equation (4.5) is used. However, Stephens (1976) showed that the asymptotic approximation given by Anderson and Darling is conservative as compared to his fitted distribution in the family of normal distributions [Stephens (1976) Table 4, p..367] and the extreme value distributions [Stephens (1977) Table 1, p. 687]. Thus, it seems reasonable that equation (4.5) is also conservative, that is, if $\alpha_n(\rho)$ is the true value for the l.h.s. of equation (4.5), then $\alpha(\rho) < \alpha_n(\rho)$.

Table 2. Type 1 Errors for Unspecified

γ	n	ρ=.25	.50	.75
1	20	1.00	1.00	.65
	50	1.00	.87	.25
	100	1.00	.50	.06
2	20	1.00	1.00	.55
	50	1.00	.74	.18
	100	1.00	.41	.04
3	20	1.00	1.00	.53
	50	1.00	.70	.16
	100	1.00	.37	.03

From Table 2, it follows that the procedure is only sensitive to ρ whenever $\rho = .75$ and $n > 50$. This observation was also supported in the simulation study.

In order to determine the sensitivity of the test to the scale parameters, the distance given by

$$D_n(s) = n \int_0^{\pi/2} \{G(\theta) - C(\theta)\}^2 \, dG(\theta) \tag{4.6}$$

where $G(\theta)$ is given in (3.2) and $C(\theta)$ is given by (3.2) whenever $\tan\theta = sr$, $s = \beta_x/\beta_y = .90(.02)1.10$. Errors in either of the scale parameters can be considered by varying s in (3.2). Table 3 contains the expected type I errors given by

$$\Pr[W_n > D_n(s)] = \alpha_n(s) \tag{4.7}$$

for different values of n, γ, and ρ.

From Table 3 one observes that the procedure appears to be resilient to errors in the scale parameter and that one might have a type I error when $\gamma = 5$, $\rho = .75$, and $n = 100$ at the 95% significance level. In addition it also appears that the results are symmetric about $s = 1$.

5. CONCLUSIONS AND SUMMARY

A procedure is outlined for determining whether a four or five parameter bivariate gamma model is appropriate. The procedure was evaluated and three

Table 3. Type 1 Errors for Misspecified Scales

γ	ρ	n	.90	.92	.94	1.06	1.08	1.10
1	0	50	1.0	1.0	1.0	1.0	1.0	1.0
		100	1.0	1.0	1.0	1.0	1.0	1.0
	.25	50	1.0	1.0	1.0	1.0	1.0	1.0
		100	.89	1.0	1.0	1.0	1.0	.87
	.50	50	1.0	1.0	1.0	1.0	1.0	1.0
		100	.67	.96	1.0	1.0	1.0	.68
	.75	50	.69	1.0	1.0	1.0	1.0	.73
		100	.37	.57	.89	.83	.60	.40
2	0	50	.98	1.0	1.0	1.0	1.0	1.0
		100	.56	.79	1.0	1.0	.80	.57
	.25	50	.78	1.0	1.0	1.0	1.0	.80
		100	.44	.64	1.0	1.0	.66	.46
	.50	50	.58	.81	1.0	1.0	.84	.61
		100	.38	.46	.74	.76	.49	.31
	.75	50	.31	.49	.78	.80	.52	.34
		100	.09	.22	.44	.46	.24	.11
3	0	50	.68	.98	1.0	1.0	1.0	1.0
		100	.36	.55	.86	.88	.57	.38
	.25	50	.55	.77	1.0	1.0	.80	.57
		100	.26	.43	.70	.72	.45	.28
	.50	50	.37	.57	.89	.92	.60	.41
		100	.14	.28	.52	.54	.30	.16
	.75	50	.17	.32	.57	.59	.34	.19
		100	.03	.10	.27	.28	.12	.04

The values of s = (.96,1.04) were omitted since the Type 1 error was 1.0 for all parameters. Likewise, whenever n=20.

different functions were evaluated in order to determine the procedure's feasibility and sensibility to the nuisance parameters. Admittedly, the evaluation is very limited and there are several limitations which would prohibit this type of procedure as an omnibus test of fit. However, the results appear to be promising to the experimentalist interested in obtaining insight into the stated problem.

There are several nonparametric procedures for testing (1.2) versus (1.3) and perhaps these are not as sensitive to the nuisance parameters. However, the proposed procedure is based upon "measuring" significant departures of the parametric distributions function which are vital to the modelers' primary objective.

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CHAPTER III

A DIFFERENTIAL EQUATIONS APPROACH TO THE MODAL LOCATION FOR A FAMILY OF BIVARIATE GAMMA DISTRIBUTIONS

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ABSTRACT

Analytical and numerical computational methods are given for determining the location of the mode as a function of the parameters of a class of the bivariate gamma distribution.

I. INTRODUCTION

Smith, Adelfang, and Tubbs (1983) derived some computational results for a family of bivariate distributions. In their paper they consider the location of the mode as a function of the shape parameters, γ_1 and γ_2, and the dependence coefficient η. The purpose of this paper is to consider this problem in greater detail. That is, the paper will consider analytical and numerical computational

methods for locating the modal values for the class of density functions given in Smith, Adelfang, and Tubbs (1983). The general density function is given by:

$$f(t_1,t_2;\gamma_1,\gamma_2,\eta) = \frac{t_1^{\gamma_1-1}t_2^{\gamma_2-1}e^{-(t_1+t_2)}}{(1-\eta)^{\gamma_1}\Gamma(\gamma_1)\Gamma(\gamma_2-\gamma_1)} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{jk} \tag{1.1}$$

$$\text{where } a_{jk} = \frac{\eta^{j+k}\Gamma(\gamma_2-\gamma_1+k)(t_1t_2)^j t_2^k}{(1-\eta)^{2j+k}\Gamma(\gamma_2+j+k)j!k!},$$

$t_1 = \beta_1 x$, $t_2 = \beta_2 y$, β_1,β_2 are scale parameters, $\gamma_2 > \gamma_1 > 1$ are shape parameters, and $0 < \eta < 1$ is associated with the correlation coefficient ρ by the equation $\eta = \rho\sqrt{\gamma_2/\gamma_1}$. We will assume without loss of generality that $\beta_1 = \beta_2 = 1$.

We will concentrate on the special case $\gamma_1 = \gamma_2 = \gamma$ of (1.1) for which the distribution function reduces to

$$f(t_1,t_2;\gamma,\eta) = \frac{(t_1t_2)^{\gamma-1}e^{-(t_1+t_2)/(1-\eta)}}{(1-\eta)^{\gamma}\Gamma(\gamma)} \sum_{j=0}^{\infty} \frac{\eta^j(t_1t_2)^j}{(1-\eta)^{2j}\Gamma(\gamma+j)j!} \tag{1.2}$$

This is the form given by Kibble (1941).

Smith and Adelfang (1981) used the above class of density functions in modeling wind gust data for the ascent flight of the Space Shuttle. A parametric model was selected in that the parameters are used to establish engineering constraints for the shuttle payload system. Thus, the modal location and value

were of interest to this particular application. The authors are not aware of any other results, either analytical or numerical, for the modal location for non-Gaussion multivariate distributions. The closest related work is in the area of density and mode estimation [3.g. Sager (1978, 1979), de Beauville (1978), and Eddy (1980)].

In Section 2 we will derive some qualitative results concerning the behavior of the modal location of (1.2) as a function of (γ, η). Section 3 presents analogous results for another borderline case $\gamma_1 = 1$, $\gamma_2 \geqq 2$ of (1.1). In Section 4 we present some numerical procedures based on the theoretical investigations of the previous sections. The general case $\gamma_2 > \gamma_1 > 1$ is considered in Section 5. We present some numerical tabulations for the modal location of (1.1) as a function of $(\gamma_1, \gamma_2, \eta)$ and consider some numerical interpolations from the borderline cases considered in Sections 2 and 3.

2. EQUAL SHAPE PARAMETERS - ANALYTICAL METHODS

Lemma 1. The function $f(t_1,t_2;\gamma,\eta)$ defined by (1.2) attains its maximum in the region $R_+^2 = \{(t_1,t_2):t_1\geqq 0,t_2\geqq 0\}$ on the line $t_1 = t_2$.

Proof: Since f is integrable and continuous over R_+^2, it is clear that f attains its maximum on R_+^2. Choose any constant $c > 0$. Let $h(t) = f(t,c-t;\gamma,\eta)$, $0 < t < c$. Then from (1.2) we have

$$h(t) = \sum_{j=0}^{\infty} K_j(\gamma,\eta)t^{\gamma+j-1}(c-t)^{\gamma+j-1},$$

where $K_j(\gamma,\eta) > 0$ is independent of t. Therefore,

$$h'(t) = \sum_{j=0}^{\infty} K_j(\gamma,\eta)(\gamma+j-1)t^{\gamma+j-2}(c-t)^{\gamma+j-2}(c-2t).$$

Since $h'(t) > 0$ for $0 < t < c/2$ and $h'(t) < 0$ for $c/2 < t < c$, h attains its maximum at $t = c/2$.. Therefore $f(t_1,t_2)$ attains its maximum along any line $t_i+t_2=c$ at the point $(c/2,c/2)$. This completes the proof.

Define $g(t;\gamma,\eta) = f(t,t;\gamma,\eta)$. Then by Lemma 1 it is sufficient to find the point on $t \geq \theta$ at which g attains its maximum value. Using (1.2) one can show that

$$g(t;\gamma,\eta) = c(\gamma,\eta)e^{-2t/(1-\eta)}h(t) \tag{2.1}$$

where $c(\gamma,\eta) = [(1-\eta)(\sqrt{\eta})^{\gamma-1}\Gamma(\gamma)]^{-1}$, and $h(t) = t^{\gamma-1}I_{\gamma-1}(p(\eta)t)$, where $I_\mu(z)$ denotes the modified Bessel function with index μ, and $p(\eta) = 2\sqrt{\eta}/(1-\eta)$.

Using [Abramowitz and Stegun (1964), Eqn. 9-6-28] it is not difficult to show that $h'(t) = p(\eta)t^{\gamma-1}\, I_{\gamma-2}(p(\eta)t)$, therefore $f(\tau,\tau;\gamma,\eta)$ is the mode at the bivariate gamma distribution given by (1.2) if and only if $g'(\tau) = 2g(\tau)/(1-\eta)$ or

$$\sqrt{\eta}I_{\gamma-2}(p(\eta)\tau) = I_{\gamma-1}(p(\eta)\tau), \tag{2.2}$$

where $p(\eta) = 2\sqrt{\eta}/(1-\eta)$.

With the aid of (2.1), we may prove the following theorem.

Theorem 1. For fixed $\gamma > 1$, let $\tau(\eta)$ denote the value at which $f(\tau(\eta), \tau(\eta);\gamma,\eta)$ is a maximum. Then τ is continuously differentiable for $0 \leq \eta < 1$ and satisfies the initial value problem

$$\tau'(\eta) = (\tau/2\eta)((2\tau - 2\gamma+3)^{-1} - (1+\eta)(1-\eta)^{-1})$$
$$\tau(0) = \gamma - 1. \qquad (2.3)$$

Proof: It is easy to show directly from (2.1) that g attains its maximum at $t = \gamma - 1$ when $\eta = 0$, so that $\tau(0) = \gamma - 1$. Furthermore $\frac{\partial g}{\partial t}$ is continuously differentiable for $0 < \eta < 1$ and computation shows that $\frac{\partial^2 g}{\partial t^2} \neq 0$ at $t = \gamma-1$ and $\eta=0$. Therefore, $\tau(\eta)$ is continuously differentiable in a neighborhood of $\eta = 0$ by the implicit function theorem. The proof will be completed by differentiating both sides of (2.1) with respect to η. After some simplification and solving for $\tau'(\eta)$ this yields

$$\tau'(\eta) = (1-\eta)I_{\gamma-2}(p(\eta)\tau)/(4\eta q(\eta)) - (1+\eta)\tau/2\eta(1-\eta) \qquad (2.4)$$

where $q(\eta) = I'_{\gamma-1}(p(\eta)\tau) - \sqrt{\eta}\, I_{\gamma-2}(p(\eta)\tau)$.

By [Abramowitz and Stegun (1964), Eqn. 9-2-26],

$$I'_{\gamma-1}(p(\eta)\tau) = I_{\gamma-2}(p(\eta)\tau) - (\gamma-1)I_{\gamma-1}(p(\eta)\tau)/p(\eta)\tau$$

$$I'_{\gamma-1}(p(\eta)\tau) = I_{\gamma-1}(p(\eta)\tau) + (\gamma-2)I_{\gamma-2}(p(\eta)\tau)/p(\eta)\tau;$$

substituting these expressions into $q(\eta)$ and using (2.2) yields after some simplification

$$q(\eta) = (1-\eta)(2\tau-2\gamma + 3)I_{\gamma-2}(p(\eta)\tau)/2\tau.$$

Substituting this expression into (2.4) completes the proof of Theorem 1.

The nonlinear differential equation (2.3) cannot be solved in general in closed form. Some numerical solutions are given

in Section 4. However (2.3) does give information regarding the qualitative and limiting behavior of $\tau(\eta)$ for $\gamma > 1$. In the special case $\gamma = 3/2$, (2.3) reduces to a linear differential equation which can be solved directly by standard methods.

Corollary 1 If $\gamma = 3/2$, then

$$\tau(\eta) = \frac{(1-\eta)}{4\sqrt{\eta}} \ln\left(\frac{1+\sqrt{\eta}}{1-\sqrt{\eta}}\right).$$

This result can also be obtained directly from (2.2) using the fact that $I_\mu(z)$ can be expressed in terms of hyperbolic functions when $\mu = \pm 1/2$.

Since the differential equation (2.3) is singular at $\eta = 0$, its numerical solution requires some additional knowledge of the behavior of $\tau(\eta)$ near $\eta = 0$. This is provided by the following corollary.

Corollary 2. The function $\tau(\eta)$ is continuously differentiable at $\eta = 0$ and satisfies

$$\tau'(0) = -(\gamma-1)/\gamma \quad , \; \gamma > 1. \tag{2.5}$$

Proof. The continuous differentiability of τ at $\eta = 0$ was considered in the proof of Theorem 1. Choose $\eta > 0$, then by the mean value theorem there is a number $\xi \in (0,\eta)$ such that

$$\tau(\eta) = \tau(0) + \eta\tau'(\xi) = \gamma-1 + \eta\tau'(\xi).$$

Substituting this expression into (2.3) and simplifying yields

$$\tau'(\eta) = \frac{(\gamma-1+\eta\tau'(\xi))(1+(1+\eta)\tau'(\xi))}{(1-\eta)(1+2\eta\tau'(\xi))}.$$

Letting $\eta \to 0$ and using the continuity of $\tau'(\eta)$ we have

$$\tau'(0) = -(\gamma-1)(1+\tau'(0)).$$

Solving this equation for $\tau'(o)$ yields (2.5)

We will write $\tau(\eta,\gamma)$ when we wish to emphasize the dependence of the modal location on γ. Theorem 1 and Corollary 2 may be used to obtain several of the qualitative and asymptotic properties of the function $\tau(\eta,\gamma)$ in the region $0 < \eta < 1$, $\gamma > 1$. These are summarized in the following theorem.

Theorem 2. The modal location function $\tau(\eta,\gamma)$ has the following properties:

(i) $\tau(\eta,\gamma)$ is a decreasing function of η for fixed $\gamma > 1$;

(ii) $\lim_{\eta\to 1} \tau(\eta,\gamma) = \max\{\gamma-\frac{3}{2},0\}$ for $\gamma > 1$;

(iii) $\tau(\eta,\gamma) - (\gamma-\frac{3}{2})$ is a decreasing function of γ for fixed $\eta \in (0,1)$ and $\gamma > 1$;

(iv) $\lim_{\gamma\to\infty} \tau(\eta,\gamma) - (\gamma-\frac{3}{2}) = \frac{1-\eta}{2(1+\eta)}$, for $0 \leq \eta \leq 1$.

Proof: We will show that $\tau'(\eta) < 0$ for $0 \leq \eta < 1$. Suppose not, then since $\tau'(0) < 0$ by Corollary 2, there is a point $\xi > 0$ such that $\tau'(\xi) = 0$ and $\tau'(\eta) < 0$ for $0 < \eta < \xi$. Let $w(\eta) = \tau(\eta) - (\gamma-\frac{3}{2})$ and $z(\eta) = \frac{1-\eta}{2(1+\eta)}$, then from (2.3) it is easy to see that

$$\tau'(\eta) = \frac{\tau(\eta)}{4\eta}\left[\frac{1}{w(\eta)} - \frac{1}{z(\eta)}\right],$$

so $\tau'(\xi) = 0$ if and only if $w(\xi) = z(\xi)$.

Let $h = w - z$. Note that $z'(\eta) = -(1+\eta)^{-2}$ so that $h'(0) = w'(0) - z'(0) = \tau'(0) + 1 > 0$ and $h(0) = w(0) - z(0) = 0$. Therefore since $h(\xi) = 0$ and $h(\eta) > 0$ for $0 < \eta < \xi$, we must have $h'(\xi) \leq 0$. However, $h'(\xi) = w'(\xi) - z'(\xi) = \tau'(\xi) - z'(\xi) = -z'(\xi) = (1+\xi)^{-2} > 0$. This contradiction proves (i). Furthermore, we have that $w(\eta) > z(\eta)$ for $0 < \eta < 1$.

We will now consider the proof of (iii). Fix $\gamma_1 > \gamma_2 > 1$ and let $f(\eta) = w(\eta,\gamma_1) - w(\eta,\gamma_2)$ where as before $w(\eta,\gamma) = \tau(\eta,\gamma) - (\gamma-\frac{3}{2})$. We wish to show that $f(\eta) < 0$ for $0 < \eta < 1$. Clearly $f(0) = 0$ and by (2.5) $f'(0) = \frac{1}{\gamma_1} - \frac{1}{\gamma_2} < 0$. Assume to obtain a contradiction that there is a point $\xi \varepsilon (0,1)$ such that $f(\xi) = 0$. If, in addition, we assume ξ is the first such point, then $f(\eta) < 0$ for $0 < \eta < \xi$ so $f'(\xi) \geq 0$. However, using (2.6) at both γ_1 and γ_2 and the fact that $w(\xi,\gamma_1) = w(\xi,\gamma_2)$ it is not difficult to show that

$$f'(\xi) = \frac{(\gamma_1-\gamma_2)}{4\eta}\left[\frac{1}{w(\xi)} - \frac{1}{z(\xi)}\right].$$

Since $\gamma_1 > \gamma_2$ and $w(\xi) > z(\xi)$, it follows that $f'(\xi) < 0$. This contradiction completes the proof of (iii).

Now we turn to the proof of (ii). First consider the case $1 < \gamma \leq 3/2$. Since τ is decreasing in η and positive for $0 \leq \eta < 1$ we know that $\tau^* = \lim_{\tau\to 1} \tau(\eta)$ exists, where the limits at 1 are always from the left. Assume to obtain a contradiction that $\tau^* > 0$. Then it is not difficult to show using (2.3) that

$$\tau'(\eta) \leqq \frac{1}{4\eta} - \frac{\tau^*(1+\eta)}{2\eta(1-\eta)}$$

Therefore, for $\frac{1}{2} \leqq \eta < 1$ we have

$$\tau'(\eta) \leqq \frac{1}{2} - \frac{\tau^*}{2(1-\eta)} .$$

Integrating both sides of this inequality from $\frac{1}{2}$ to η yields

$$\tau(\eta) \leqq \tau(\tfrac{1}{2}) + \frac{1}{2} + \frac{\tau^*}{2} \ln(1-\eta)$$

for $\frac{1}{2} \leqq \eta < 1$. However, this implies that $\tau(\eta) \to -\infty$ as $\eta \to 1$, a contradiction.

The case $\gamma > \frac{3}{2}$ follows easily from (iii) and the proof of (i) because for $\gamma \geqq \frac{3}{2}$

$$z(\eta) \leqq \tau(\eta) - (\gamma-\tfrac{3}{2}) \leqq \tau(\eta,\tfrac{3}{2})$$

and both $z(\eta)$ and $\tau(\eta,\frac{3}{2})$ approach zero as $\eta \to 1$.

Finally, we consider the proof of (iv). Let $u(\eta,\gamma) = w(\eta,\gamma) - z(\eta)$ for $0 \leqq \eta \leqq 1$ and $\gamma \geqq \frac{3}{2}$. From the proof of (i) we know that $u(\eta,\gamma) \geqq 0$. From (2.6) we obtain

$$w'(\eta,\gamma) = \frac{1}{4\eta}\left[1 - \frac{w(\eta,\gamma)}{z(\eta)}\right] + \frac{(\gamma-\frac{3}{2})}{4\eta}\left[\frac{z(\eta) - w(\eta,\gamma)}{w(\eta,\gamma)z(\eta)}\right]$$

so that

$$w'(\eta,\gamma) \leqq -(\gamma-\tfrac{3}{2})u(\eta,\gamma).$$

Therefore,

$$u'(\eta,\gamma) = w'(\eta,\gamma) - z'(\eta) \leqq -(\gamma-\tfrac{3}{2})u(\eta,\gamma) + 1.$$

From this inequality we obtain

$$\frac{d}{d\eta}\left[u(\eta,\gamma)e^{(\gamma-\frac{3}{2})\eta}\right] \leqq e^{(\gamma-\frac{3}{2})\eta}$$

$$u(\eta,\gamma)e^{(\gamma-\frac{3}{2})\eta} - u(0,\gamma) \leqq \frac{1}{(\gamma-\frac{3}{2})}\left[e^{(\gamma-\frac{3}{2})\eta} - 1\right].$$

Therefore

$$0 \leqq u(\eta,\gamma) \leqq \frac{1}{(\gamma-\frac{3}{2})}.$$

This implies that $u(\eta,\gamma) \to 0$ as $\gamma \to \infty$ and completes the proof of Theorem 2.

3. UNEQUAL SHAPE PARAMETERS--THE CASE $\gamma_1 = 1$

In this section we consider another "borderline" case of the general bivariate gamma distribution, the case $\gamma_1 = 1$. For technical reasons we will limit our discussion to the range $\gamma_2 \geqq 2$ and for brevity let $\gamma_2 = \gamma$. Then the function given by (1.1) reduces to

$$f(t_1,t_2;1,\gamma,\eta) = \frac{t_2^{\gamma-1}e^{-s_2}}{(1-\eta)\Gamma(\gamma-1)}\sum_{j=0}^{\infty} e^{-s_1}\, c_j\, \frac{s_1^j}{j!} \tag{3.1}$$

where

$$c_j = \sum_{k=0}^{\infty} s_3^{j+k}\,\frac{\Gamma(\gamma+k-1)}{k!\,\Gamma(\gamma+j+k)}\ ,\ j = 0,1,2,\ldots, \tag{3.2}$$

and where $s_1 = \frac{t_1}{1-\eta}$, $s_2 = \frac{t_2}{1-\eta}$, and $s_3 = \eta s_2$.

The following lemma allows us to restrict our attention to the line $t_1 = 0$.

Lemma 2. The function $f(t_1,t_2;1,\gamma,\eta)$ given by (3.1) for $\gamma \geq 2$, takes on its maximum value in the region $t_1 \geq 0$, $t_2 \geq 0$ on the line $t_1 = 0$.

Proof: Since f is continuous and integrable in the first quadrant, we know it takes on its maximum value at some point (t_1^*, t_2^*). We will prove that $t_1^* = 0$ by showing that for any fixed $t_2 > 0$, $f(t_1,t_2)$ is a decreasing function of t_1. This is equivalent to showing that the function

$$g(s) = \sum_{j=0}^{\infty} e^{-s} c_j \frac{s^j}{j!}$$

is a decreasing function on $s \geq 0$ where c_j is given by (3.2). Note that

$$g'(s) = -e^{-s} \sum_{j=0}^{\infty} c_j \frac{s^j}{j!} + e^{-s} \sum_{j=0}^{\infty} c_j \frac{j s^{j-1}}{j!}$$

$$= -e^{-s} \sum_{j=0}^{\infty} c_j \frac{s^j}{j!} + e^{-s} \sum_{j=0}^{\infty} c_{j+1} \frac{s^j}{j!}$$

$$= -e^{-s} \sum_{j=0}^{\infty} \frac{s^j}{j!} (c_j - c_{j+1}).$$

Therefore $g'(s) < 0$ for $s \geq 0$ if $c_{j+1} < c_j$ for $j = 0,1,2,\ldots$. To this end note that

$$c_{j+1} = \sum_{k=0}^{\infty} s_3^{j+1+k} \frac{\Gamma(\gamma+k-1)}{k!\Gamma(\gamma+j+1+k)}$$

$$= \sum_{k=1}^{\infty} s_3^{j+k} \frac{\Gamma(\gamma+k-2)}{(k-1)!\Gamma(\gamma+j+k)}$$

$$= \sum_{k=1}^{\infty} s_3^{j+k} \frac{\Gamma(\gamma+k-1)}{k!\Gamma(\gamma+j+k)} \cdot \frac{k}{\gamma+k-2} < c_j$$

since $\gamma \geq 2$ implies that $\frac{k}{\gamma+k-2} \leq 1$ for $k = 1,2,3,\ldots$. This completes the proof of Lemma 2.

According to the preceding lemma, the mode of the bivariate gamma distribution in this case is the point $(0,\mu)$ where μ is the point on $t \geq 0$ where the following function is a maximum:

$$g(t) = t^{\gamma-1} e^{-t/(1-\eta)} h(t)$$

where

$$h(t) = \sum_{k=0}^{\infty} \left(\frac{\eta t}{1-\eta}\right)^k \frac{\Gamma(\gamma+k-1)}{k!\,\Gamma(\gamma+k)} = \sum_{k=0}^{\infty} \left(\frac{\eta t}{1-\eta}\right)^k \frac{1}{k!(\gamma+k-1)} .$$

Note that

$$\frac{d}{dt}\left(\left(\frac{\eta t}{1-\eta}\right)^{\gamma-1} h(t)\right) = \frac{d}{dt}\left(\sum_{k=0}^{\infty} \left(\frac{\eta t}{1-\eta}\right)^{\gamma+k-1} \frac{1}{k!(\gamma+k-1)}\right)$$

$$= \frac{\eta}{1-\eta} \sum_{k=0}^{\infty} \left(\frac{\eta t}{1-\eta}\right)^{\gamma+k-2} \frac{1}{k!}$$

$$= \frac{\eta}{1-\eta} \left(\frac{\eta t}{1-\eta}\right)^{\gamma-2} e^{\eta t/(1-\eta)} .$$

Therefore,

$$h(t) = \left(\frac{1-\eta}{\eta t}\right)^{\gamma-1} \frac{\eta}{1-\eta} \int_0^t \left(\frac{\eta s}{1-\eta}\right)^{\gamma-2} e^{\eta s/(1-\eta)}\, ds$$

$$= \frac{1}{t^{\gamma-1}} \int_0^t s^{\gamma-2} e^{\eta s/(1-\eta)}\, ds$$

so that the function we wish to maximize is

$$g(t) = e^{-t/(1-\eta)} \int_0^t s^{\gamma-2} e^{\eta s/(1-\eta)}\, ds. \tag{3.3}$$

Lemma 4. Let $\mu(\eta)$, or when necessary $\mu(\eta,\gamma)$, denote the value for which $f(0,\mu(\eta,\gamma);1,\gamma,\eta)$ is a maximum where f is defined by (3.1) and (3.2). Then $\mu(0) = \gamma-1$, $\mu(\eta) \geq \gamma-2$ for $0 \leq \eta < 1$, and satisfies the equation

$$g(\mu) = (1-\eta)\mu^{\gamma-2}e^{-\mu}, \quad 0 \leq \eta < 1, \tag{3.4}$$

where g is defined by (3.3).

Proof: It is easy to see from (3.3) that g attains its maximum on $[0,\infty)$ at a point $t^* > 0$ for which $g'(t^*) = 0$ and $g''(t^*) \leq 0$. Differentiating (3.3) we obtain

$$g'(t) = -\frac{1}{1-\eta}\, g(t) + t^{\gamma-2}e^{-t}$$

and

$$g''(t) = -\frac{1}{1-\eta}\, g'(t) + t^{\gamma-3}e^{-t}(\gamma-2-t).$$

Therefore $g'(\mu) = 0$ implies (3.4) and $g''(\mu) \leq 0$ implies that $\mu \geq \gamma - 2$. Since when $\eta = 0$, $g(t) = e^{-t}\frac{t^{\gamma-1}}{\gamma-1}$ it is easy to see that $\mu(0) = \gamma-1$. This completes the proof of the lemma.

With the aid of these preliminaries we may prove the following theorem in the spirit of Theorem 1.

Theorem 3. For fixed $\gamma \geq 2$, let $\mu(\eta)$ denote the value of which $f(0,\mu(\eta);\ 1,\gamma,\eta)$ is a maximum. Then μ is continuously differentiable on $0 \leq \eta < 1$, $\mu'(0) = -1 + \frac{1}{\gamma}$, and on $0 < \eta < 1$ μ satisfies the initial value problem

$$\mu'(\eta) = -\frac{\mu(\mu - (\gamma-1) + \eta)}{\eta(1-\eta)(\mu-(\gamma-2))};$$

$$\mu(0) = \gamma-1. \tag{3.5}$$

Proof: As in the proof of Theorem 1 the continuous differentiability of μ in a neighborhood of $\eta=0$ may be proved by applying the implicit function theorem to (3.4). This differentiability will be extended to all of $[0,1)$ by proving that (3.5) holds. Let $g(t,\eta)$ denote the function defined by (3.3) and let g_t and g_η denote its partial derivatives with respect to t and η, respectively. Then differentiating both sides of (3.4) with respect to η we obtain

$$g_t(\mu,\eta)\mu' + g_\eta(\mu,\eta) = (1-\eta)e^{-\mu}\mu^{\gamma-3}(\gamma-2-\mu)\mu' - e^{-\mu}\mu^{\gamma-2}. \tag{3.6}$$

By definition $g_t(\mu,\eta) = 0$ and direct differentiation of (3.4) and integration by parts yields for $0 < \eta < 1$ that

$$g_\eta(t,\eta) = -\frac{t}{(1-\eta)^2} g(t,\eta)$$

$$+ \frac{1}{(1-\eta)^2} e^{-t/(1-\eta)} \int_0^t s^{\gamma-1} e^{\eta s/(1-\eta)} ds$$

$$= -\frac{t}{(1-\eta)^2} g(t,\eta)$$

$$+ \frac{1}{(1-\eta)^2} e^{-t/(1-\eta)} [\frac{1-\eta}{\eta} t^{\gamma-1} e^{\eta t/(1-\eta)}$$

$$- \frac{1-\eta}{\eta} (\gamma-1) \int_0^t s^{\gamma-2} e^{\eta s/(1-\eta)} ds]$$

$$= -\frac{t}{(1-\eta)^2}\, g(t,\eta) + \frac{1}{\eta(1-\eta)}\, t^{\gamma-1}e^{-1} - \frac{\gamma-1}{\eta(1-\eta)}\, g(t,\eta).$$

Therefore, using (3.4) we obtain

$$g_\eta(\mu,\eta) = -\frac{\mu}{1-\eta}\,\mu^{\gamma-2}e^{-\mu} + \frac{1}{\eta(1-\eta)}\,\mu^{\gamma-1}e^{-\mu}$$

$$- \frac{\gamma-1}{\eta}\,\mu^{\gamma-2}e^{-\mu}$$

$$= \mu^{\gamma-2}e^{-\mu}\left(-\frac{\mu}{1-\eta} + \frac{\mu}{\eta(1-\eta)} - \frac{\gamma-1}{\eta}\right)$$

$$= \mu^{\gamma-2}e^{-\mu}\,\frac{\mu-(\gamma-1)}{\eta}.$$

Substituting this expression into (3.6) and simplifying yields (3.5). For $\eta = 0$, an easy calculation shows that

$$g_\eta(t,0) = -\frac{1}{\gamma(\gamma-1)}\, e^{-t}t^\gamma$$

from which substitution into (3.6) with $\eta = 0$ and $\mu = \gamma - 1$ shows that $\mu'(0) = -1 + \frac{1}{\gamma}$. This completes the proof of Theorem 3.

The following corollary exploits the fact that (3.5) reduces to a linear differential equation when $\gamma = 2$.

Corollary 3. If $\gamma = 2$, then

$$\mu(\eta) = \frac{1-\eta}{\eta}\ln\left(\frac{1}{1-\eta}\right),\quad 0 < \eta < 1.$$

Proof. For $\gamma = 2$ equation (3.5) reduces to

$$\mu' + \frac{\mu}{\eta(1-\eta)} = \frac{1}{\eta}$$

which is easily solved in closed form by standard methods to show the desired result. This result is also easily derived directly from (3.4).

It is interesting to note that the translated modal location function $v(\eta) = \mu(\eta) - (\gamma-2)$ satisfies the differential equation

$$v'(\eta) = \frac{v(\eta) + \gamma-2}{\eta}\left(\frac{1}{v(\eta)} - \frac{1}{1-\eta}\right)$$

$$v(0) = 1,\ v'(0) = -1 + \frac{1}{\gamma},$$

whereas the translated modal location function $w(\eta) = \tau(\eta) - (\gamma-\frac{3}{2})$ of Section 2 satisfies the analogous differential equation

$$w'(\eta) = \frac{w(\eta) + \gamma - \frac{3}{2}}{4\eta}\left(\frac{1}{w(\eta)} - \frac{2(1+\eta)}{1-\eta}\right)$$

$$w(0) = \frac{1}{2}\ ,\ w'(0) = -1 + \frac{1}{\gamma}.$$

For this reason μ behaves in a manner similar to τ. Its properties are stated in the following theorem. Since the proof of this theorem is entirely analogous to the proof of Theorem 2, it is omitted.

Theorem 4. The modal location function $\mu(\eta,\gamma)$ has the following properties:

(i) $\mu(\eta,\gamma)$ is a decreasing function of η for fixed $\gamma \geq 2$;

(ii) $\lim_{\eta\to 1}\mu(\eta,\gamma) = \gamma - 2$ for $\gamma \geq 2$;

(iii) $\mu(\eta,\gamma) - (\gamma-2)$ is a decreasing function of γ for $\gamma \geq 2$ and fixed $\eta \in (0,1)$;

(iv) $\lim_{\gamma\to\infty}(\mu(\eta,\gamma) - (\gamma-2)) = 1-\eta$ for $0 \leq \eta \leq 1$.

4. NUMERICAL RESULTS

In this section we present some quantitative results based on the results of the previous sections. Table 1 shows the value of the modal location function for equal shape parameters $\tau(\eta,\gamma)$ for various values of η and γ. Table 2 shows values of the translated modal location function $w(\eta,\gamma) = \tau(\eta,\gamma) - (\gamma-\frac{3}{2})$. This table illustrates the qualitative behavior of this function derived in Theorem 2. The limiting values of $\eta = 1$ and $\gamma = \infty$ are taken from Theorem 2.

The values in Tables 1 and 2 were computed using Theorem 1. Specifically, a fourth-order Runge-Kutta algorithm was used to compute an approximate solution of the differential equation (2.3) on the interval $0 \leq \eta < 1$ for each specified value of γ. Since equation (2.3) is singular at $\eta = 0$, Corollary 2 was used to replace the initial condition $\tau(0) = \gamma-1$ by the approximate initial condition

$$\tau(h) = \gamma-1 - \left(\frac{\gamma-1}{\gamma}\right) h$$

where h is the step size of the numerical method. Figure 1 shows the data of Table 2 in graphical form and illustrates the behavior of the function $w(\eta,\gamma)$ derived in Theorem 2.

Tables 3 and 4 show the corresponding results for the modal location function $\mu(\eta,\gamma)$ for the case $\gamma_1 = 1$, $\gamma_2 = \gamma$ and its associated translate $v(\eta,\gamma) = \mu(\eta,\gamma) - (\gamma-2)$. These tables were computed by the same methods as Tables 1 and 2 except using the results of Section 3. Figure 2 illustrates the qualitative behavior of the function $v(\eta,\gamma)$ as indicated by Theorem. 4.

Note that the differential equations (2.3) and (3.5) allow the modal location to be computed recursively in η for a fixed value of γ as a dynamic process in a time scale measured by the modified correlation coefficient η. Error in the computation is introduced through the discretization of this continuous evolutionary process. A more conventional computation of the modal location would require an independent calculation for each value of η with error introduced through the truncation of the series representation (1.2) of the distribution function. This error becomes particularly troublesome as $\eta \to 1$.

Table 1. Selected values of the modal location function $\tau(\eta,\gamma)$ for equal shape parameters.

$\gamma \backslash \eta$	0	.1	.3	.5	.7	.9	1
1.1	.1000	.0908	.0720	.0525	.0322	.0110	.0000
1.3	.3000	.2765	.2268	.1724	.1117	.0412	.0000
1.5	.5000	.4660	.3931	.3116	.2169	.0958	.0000
2.0	1.0000	.9491	.8412	.7277	.6127	.5279	.5000
3.0	2.0000	1.9334	1.8034	1.6862	1.5939	1.5268	1.5000
10.0	9.0000	8.9152	8.7754	8.6697	8.5891	8.5624	8.5000

Table 2. Selected values of the translated modal location function $w(\eta,\gamma)$ for equal shape parameters.

$\gamma \backslash \eta$	0	.1	.3	.5	.7	.9	1
1.1	.5000	.4908	.4720	.4525	.4322	.4110	.4000
1.3	.5000	.4765	.4268	.3724	.3117	.2412	.2000
1.5	.5000	.4660	.3931	.3116	.2169	.0958	.0000
2.0	.5000	.4491	.3412	.2277	.1127	.0279	.0000
3.0	.5000	.4334	.3034	.1862	.0939	.0268	.0000
10.0	.5000	.4152	.2754	.1697	.0891	.0264	.0000
∞	.5000	.4091	.2692	.1667	.0882	.0263	.0000

Table 3. Selected values of the modal location function $\mu(\eta,\gamma)$ for $\gamma_1 = 1$, $\gamma_2 = \gamma \geqq 2$.

$\gamma \backslash \eta$	0	.1	.3	.5	.7	.9	1
2.0	1.0000	.9482	.8322	.6931	.5160	.2558	.0000
3.0	2.0000	1.9310	1.7767	1.5936	1.3702	1.1111	1.0000
10.0	9.0000	8.9085	8.7170	8.5155	8.3081	8.1011	8.0000

Table 4. Selected values of the translated modal location function $\nu(\eta,\gamma)$ for $\gamma_1 = 1$, $\gamma_2 = \gamma \geqq 2$.

$\gamma \backslash \eta$	0	.1	.3	.5	.7	.9	1
2.0	1.0000	.9482	.8322	.6931	.5160	.2558	.0000
3.0	1.0000	.9310	.7767	.5936	.3702	.1111	.0000
10.0	1.0000	.9085	.7170	.5155	.3081	.1011	.0000
∞	1.0000	.9000	.7000	.5000	.3000	.1000	.0000

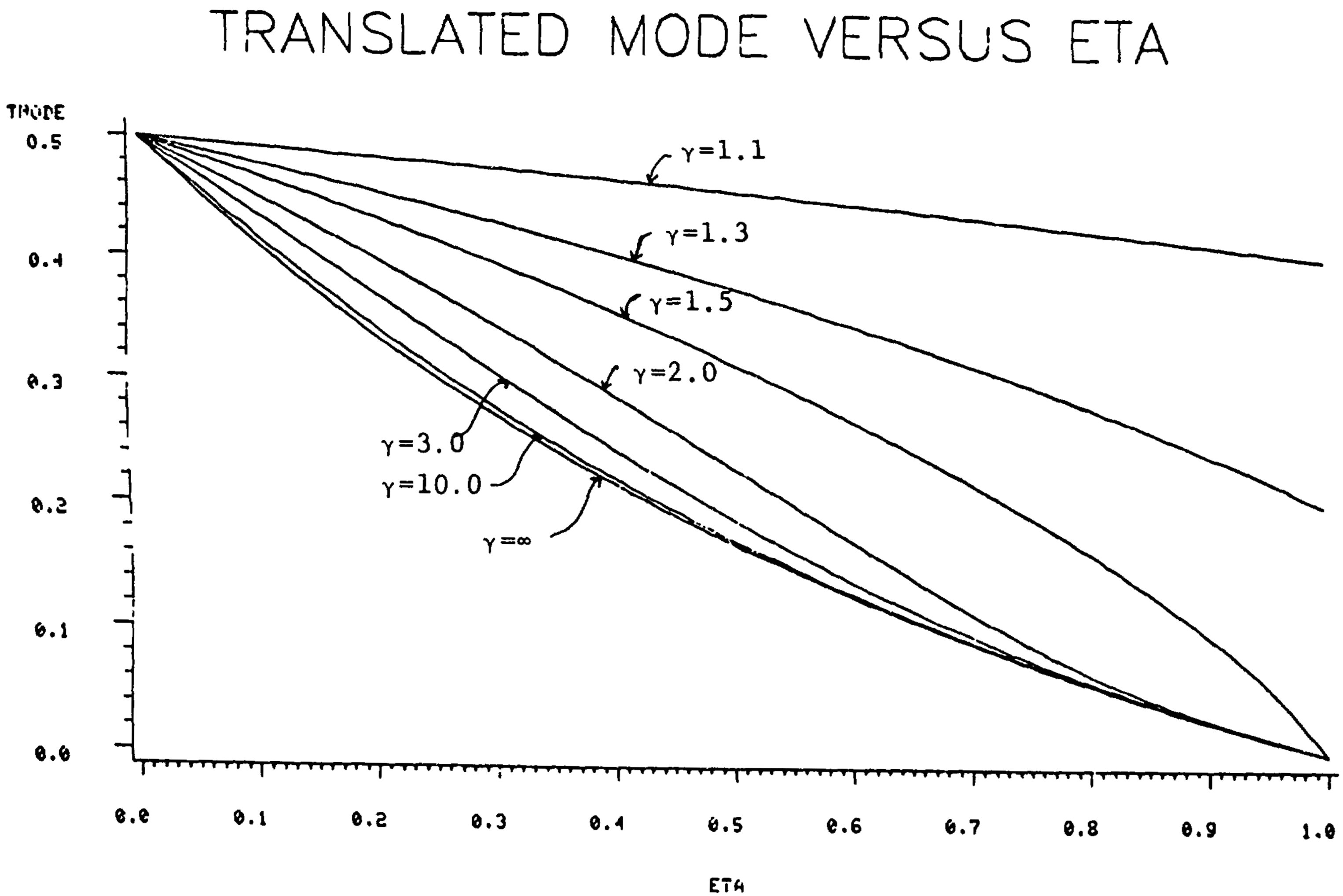

Figure 1. Qualitative behavior of the translated modal location function $w(\eta,\gamma)$ for equal shape parameters.

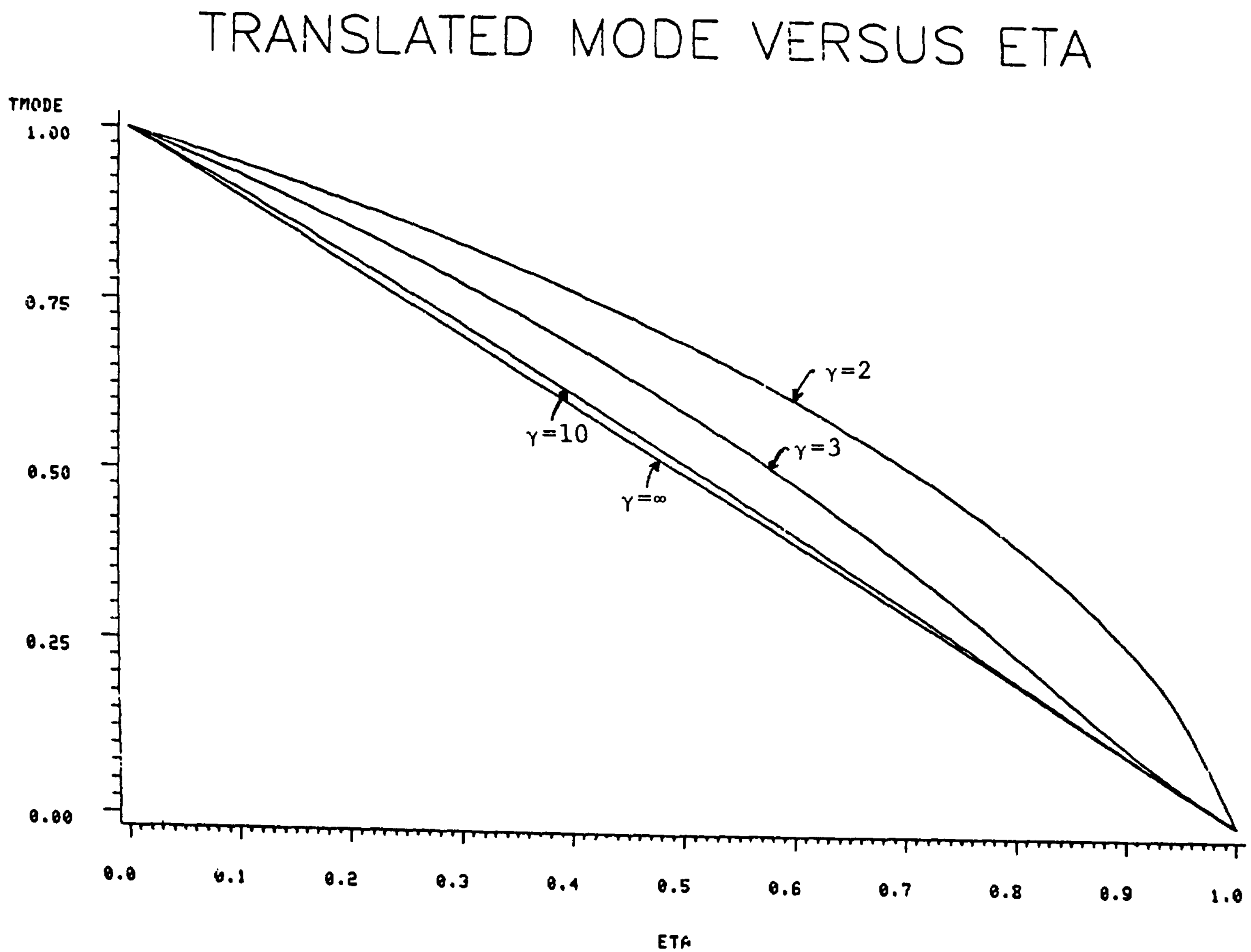

Figure 2. Qualitative behavior of the translated modal location function v(...) for $\gamma_1=1$, $\gamma_2=\gamma\geq 2$.

5. UNEQUAL SHAPE PARAMETERS

In this section we briefly consider the mode of the general bivariate gamma distribution given by (1.1). By setting the partial derivatives of $f(t_1,t_2;\gamma_1,\gamma_2,\eta)$ with respect to t_1 and t_2 equal to zero, one finds that f attains its maximum at the point (t_1,t_2) whose coordinates satisfy

$$t_1 = \frac{1-\eta}{S} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} a_{jk}(\gamma_1+j-1) \tag{5.1}$$

and

$$t_2 = \frac{1-\eta}{S} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} a_{jk}(\gamma_2+j+k-1) \tag{5.2}$$

where $S = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} a_{jk}$

and a_{jk} given as in (1.1) depends on t_1 and t_2.

Table 5 shows selected values of the modal location for the case $\gamma_2 = 3$. They were computed by truncating each of the series in (5.1) and (5.2) to about fifty terms and simultaneously iterating on these equations until an approximate solution is obtained. These computations become unreliable as $\eta \to 1$ and the truncation error becomes unacceptable.

Figure 3 gives a graphical representation of the change in modal location with η and γ_1 for fixed $\gamma_2 = 3$. It is interesting to note for a fixed η the extent to which the modal location may be approximated by linear interpolation between the borderline cases discussed in Sections 2 and 3.

More specifically, we have the empirical approximations

$$t_1 = \frac{\gamma_1 - 1}{\gamma_2 - 1} \tau(n, \gamma_2) \tag{5.3}$$

and

$$t_2 = \mu(n, \gamma_2) + \frac{\gamma_1 - 1}{\gamma_2 - 1} (\tau(n, \gamma_2) - \mu(n, \gamma_2)) \tag{5.4}$$

where τ and μ are as defined in Sections 2 and 3, respectively. This empirical relationship is a subject for further investigation.

Table 5. Location of the mode using (5.1) and (5.2) with $\gamma_2 = 3$. Approximate values computed using (5.3) and (5.4) are denoted by *. When these values are equal to two decimal places, only one is given.

$\eta \backslash \gamma_1$	1	1.5	2	2.5	3
0	(0,2.00)	(.50,2.00)	(1.00,2.00)	(1.50,2.00)	(2.00,2.00)
.25	(0,1.82)	(.46,1.82)	(.92,1.83)	(1.38,1.83)	(1.84,1.84)
.50	(0,1.59)	(.42,1.62)	(.84,1.64)	(1.26,1.66)	(1.69,1.69)
.75	(0,1.31)	(.40,1.38) (.39,1.37)*	(.80,1.44) (.70,1.44)*	(1.19,1.51) (1.18,1.51)*	(1.58,1.58)
.85	(0,1.18)	(.41,1.26) (.39,1.27)*	(.80,1.36) (.77,1.36)*	(1.17,1.45) (1.16,1.45)*	(1.54,1.54)

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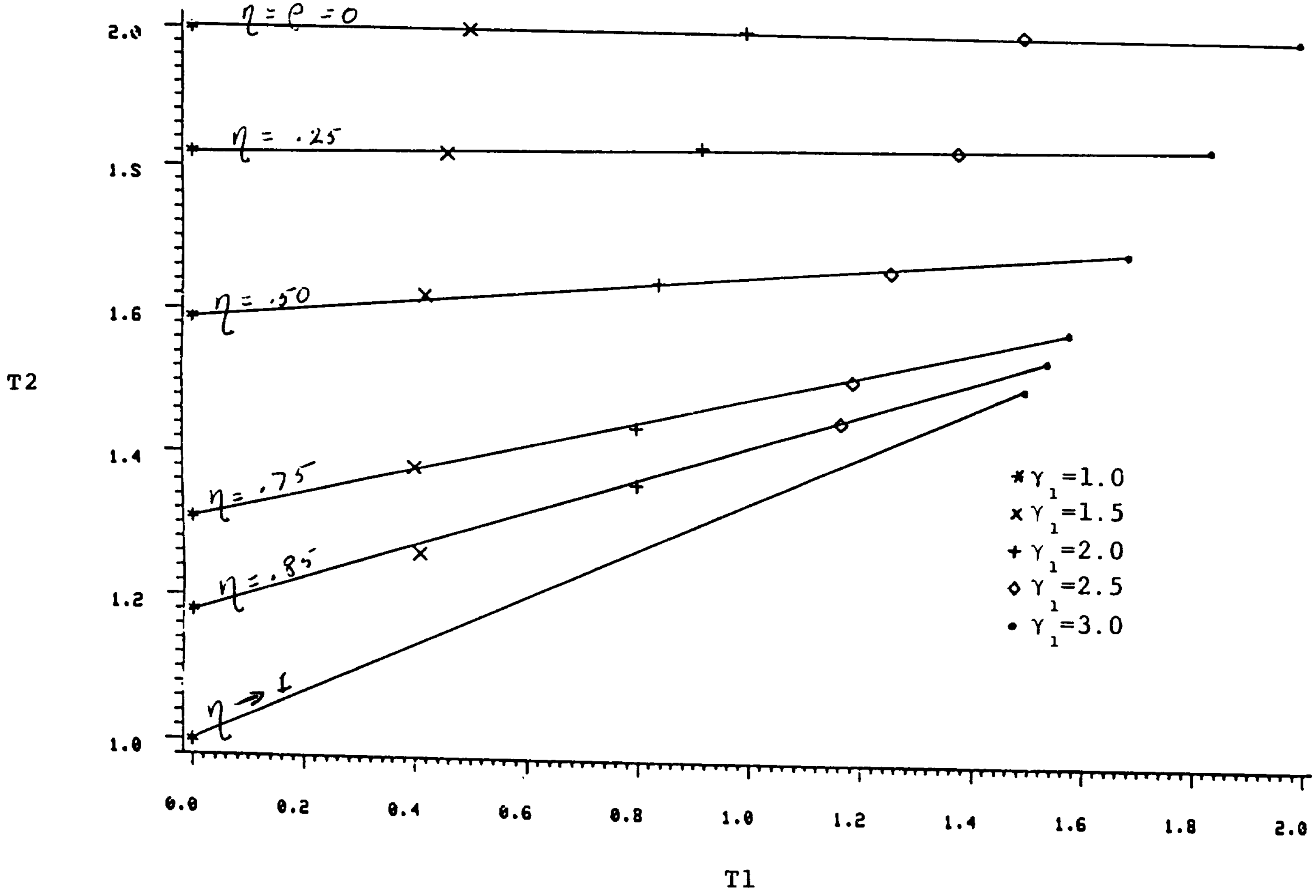

Figure 3. Graphical representation of the values in Table 5.

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CHAPTER IV

ANALYSIS OF WIND GUST DATA

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ABSTRACT

This paper summarizes the analysis of wind gust data using statistical and mathematical procedures which were developed for the bivariate gamma distribution.

1. INTRODUCTION

Adelfang and Smith (1981) discuss the use of the gamma distribution in modeling gust data at Cape Canaveral, Florida. Smith and Adelfang (1981) treated gust amplitude and length scale as the variables of the bivariate gamma distribution. Smith, Adelfang, and Tubbs (1983) presented some useful analytical and computational results for a class of the bivariate gamma and applied some of these results to the wind gust data. The purpose of this paper is to analyze the wind gust data using some additional analytical results obtained for the bivariate gamma distribution.

2. DATA

The data used in this paper consists of absolute gust

magnitude and gust length for both the zonal and meridional components. The 150 wind profiles were filtered using the band pass filter for wavelengths within 420-2470 meter band. Data were available for the reference altitudes: 4Km, 6Km, 8Km, 10Km, 12Km, and 14Km. The data was paired into bivariate components for both the zonal and meridional components, denoted by the pairs (Au,Lu) and Av,Lv), respectively.

3. ANALYTICAL PROCEDURES

The data were partitioned according to reference altitudes, then the 150 observations were analyzed using both univariate and multivariate techniques. Simple descriptive univariate techniques were generated using PROC UNIVARIATE in SAS. These procedures were used to help in the assessment of the marginal distribution. The multivariate descriptive procedures consisted of bivariate scatter plots and contour plots.

Goodness of fit tests consisted of a univariate test for marginal normality generated by SAS, two tests for bivariate normality as discussed in Meredith and Tubbs (1981), and a bivariate test for the gamma distribution. The latter procedure is a bivariate Chi-square type test which uses the computational results for the distribution function as presented in Smith, Adelfang, and Tubbs (1983).

Parameter estimates for the bivariate gamma distribution were evaluated. These estimates were then used in

generating the three-dimensional bivariate gamma density function plots and the modal locations were estimated using the results given by Brewer, Tubbs, and Smith (1983).

4. RESULTS

The results are summarized in Tables 1-7. Additional results are given in Appendices A and B.

Tables 1-6 summarize the results for both the test of fit and the parameter estimates for the bivariate gamma distribution. There are two main tests for bivariate normality and both of these are discussed in Meredith and Tubbs (1981). The first is a procedure proposed by Rincon-Gallardo et al. (1979). Since this procedure transforms the data to a univariate test for uniformity three different tests for uniformity are used. The second test for bivariate normality is based upon a procedure proposed by Cox and Small (1978).

The bivariate test for the gamma distribution is a Chi-square type test of fit. Thus, this procedure has the usual difficulties of selecting the number of cells and cell location that are associated with this type of test. In the interest of time and space a fixed procedure was applied for all the data sets. Namely, the marginal distributions were partitioned according to the .05, .25, .50, .75, and .90 quantiles based upon the gamma parameter estimates. This particular choice affected the results for some of the data sets, however, it seemed a reasonable global choice.

The univariate gamma parameters were estimated using a maximum likelihood procedure presented by Greenwood and Durand (1960) and discussed in Tubbs and Brewer (1981).

Appendix A contains the results for the univariate descriptive statistics. Appendix B contains plots for each data set. The density functions were generated using the gamma parameter estimates. The contour plots are level slices of the density function and are not equal probability contours. The location of the mode is denoted by the symbol + and this value is computed using the analytical results given in Brewer et al. (1983). Table 7 summarizes the results for the modal location.

Table 1. Summary for Wind Gust Statistic
Using Band Filter 420-2470 Altitude = 4 Km.

Multivariate Test			(Au,Lu)	(Av,Lv)
Normality	Cramer-Von Mises		.2062	.2144
	Watson's U^2		.2023*	.2058*
	K - S		.0618	.0551
	Cox		11.45**	26.07***
Gamma	Chi-Square		33.3	53.00***

Univariate Test				
Normality	Au		.077*	
	Lu		.068	
	Av		.090**	
	Lv		.077*	

Parameter Estimates		$\hat{\gamma}$	$\hat{\beta}$	$\hat{\rho}$
	Au	3.402	2.430	.2280
	Lu	5.275	.007	
	Av	2.808	1.891	.3415
	Lv	4.429	.006	

* denotes that test is significant at .05 level.
** denotes that test is significant at .01 level.
*** denotes that test is significant at .001 level.

Table 2. Summary for Wind Gust Statistic
Using Band Filter 420-2470 Altitude = 6 Km.

Multivariate Test		(AuLu)	(Av,Lv)
Normality	Cramer-Von Mises	.3806	.2942
	Watson's U^2	.2411*	.2208*
	K - S	.0897	.0623
	Cox	24.4***	31.8***
Gamma	Chi-Square	72.03***	57.14***

Univariate Test		
Normality	Au	.047
	Lu	.081*
	Av	.054
	Lv	.072

Parameter Estimates	$\hat{\gamma}$	$\hat{\beta}$	$\hat{\rho}$
Au	2.577	1.916	.4217
Lu	4.102	.005	
Av	3.168	2.030	.2506
Lv	4.895	.005	

* denotes that test is significant at .05 level.
** denotes that test is significant at .01 level.
*** denotes that test is significant at .001 level.

Table 3. Summary for Wind Gust Statistic
Using Band Filter 420-2470 Altitude = 8 Km.

Multivariate Test			(Au,Lu)	(Av,Lv)
Normality	Cramer-Von Mises		1.090***	.490*
	Watson's U^2		.721***	.409***
	K - S		.102*	.083
	Cox		10.67**	8.67*
Gamma	Chi-Square		34.74	52.57**
Univariate Test				
Normality	Au		.114**	
	Lu		.108**	
	Av		.107**	
	Lv			
Parameter Estimates		$\hat{\gamma}$	$\hat{\beta}$	$\hat{\rho}$
	Au	3.023	2.113	.3396
	Lu	4.149	.005	
	Av	2.922	1.811	.4253
	Lv	4.614	.005	

* denotes that test is significant at .05 level.
** denotes that test is significant at .01 level.
*** denotes that test is significant at .001 level.

Table 4. Summary for Wind Gust Statistic
Using Band Filter 420-2470 Altitude = 10 Km.

Multivariate Test		(Au,Lu)	(Av,Lv)
Normality	Cramer-Von Mises	.129	.753**
	Watson's U^2	.126	.469**
	K - S	.052	.104*
	Cox	8.23*	8.58*
Gamma	Chi-Square	45.04**	36.29

Univariate Test		
Normality	Au	.073*
	Lu	.063
	Av	.073*
	Lv	.109**

Parameter Estimates		$\hat{\gamma}$	$\hat{\beta}$	$\hat{\rho}$
	Au	3.041	1.909	.4345
	Lu	5.203	.006	
	Av	2.522	1.300	.3890
	Lv	4.543	.005	

* denotes that test is significant at .05 level.
** denotes that test is significant at .01 level.
*** denotes that test is significant at .001 level.

Table 5. Summary for Wind Gust Statistic
Using Band Filter 420-2470 Altitude = 12 Km.

Multivariate Test		(Au,Lu)	(Av,Lv)
Normality	Cramer-Von Mises	.465*	.398
	Watson's U^2	.391**	.329**
	K - S	.077	.076
	Cox	25.04***	23.70***
Gamma	Chi-Square	52.54**	41.24*

Univariate Test		
Normality	Au	.116**
	Lu	.112**
	Av	.042**
	Lv	.085**

Parameter Estimates		$\hat{\gamma}$	$\hat{\beta}$	$\hat{\rho}$
	Au	2.155	.969	.2983
	Lu	4.113	.005	
	Av	2.612′	1.051	.3066
	Lv	4.023	.005	

* denotes that test is significant at .05 level.
** denotes that test is significant at .01 level.
*** denotes that test is significant at .001 level.

Table 6. Summary for Wind Gust Statistic
Using Band Filter 420-2470 Altitude = 14 Km.

Multivariate Test

Normality	Cramer-Von Mises		1.359***	.616*
	Watson's U^2		..647***	.357**
	K - S		.134**	.084
	Cox		14.6***	17.9***
Gamma	Chi-Square	48	48.29**	42.97*

Univariate Test

Normality	Au	.090**
	Lu	.096**
	Av	.085**
	Lv	.061

Parameter Estimates

	$\hat{\gamma}$	$\hat{\beta}$	$\hat{\rho}$
Au	3.803	1.549	.2420
Lu	4.725	.005	
Av	3.324	1.161	.3171
Lv	5.057	.006	

* denotes that test is significant at .05 level.
** denotes that test is significant at .01 level.
*** denotes that test is significant at .001 level.

Table 7. Modal Location

Altitude	Variables	Method* I		Method* II	
4000	(Au,Lu)	.939	581.9	.939	581.7
	(Av,Lv)	.874	516.7	.873	516.7
6000	(Au,Lu)	.747	539.6	.745	539.4
	(Av,Lv)	1.008	729.4	1.007	729.4
8000	(Au,Lu)	.875	571.0	.874	570.8
	(Av,Lv)	.965	639.8	.960	639.8
10000	(Au,Lu)	.984	624.8	.977	625.2
	(Av,Lv)	1.064	624.8	1.058	624.8
12000	(Au,Lu)	1.083	556.8	1.080	556.8
	(Av,Lv)	1.402	546.2	1.400	546.2
14000	(Au,Lu)	1.713	705.4	1.713	705.4
	(Av,Lv)	1.868	626.0	1.865	626.0

* Method I Truncation of a double series.
Method II Interpolation.

5. SUMMARY

The data sets are discussed according to reference altitude.

4 Km. The normal or the gamma are not rejected for the zonal (u) components. As discussed in Meredith and Tubbs (1981) the Cox and Small procedure is sensitive to symmetry and is not recommended for this data. The gamma distribution was rejected for the v-component and normality was accepted. However, marginal normality was rejected at the .01 level for the absolute gust magnitude (Av).

6 Km. The bivariate gamma was rejected in both wind components and normality was not rejected.

8 Km. Normality was rejected for both wind components. The bivariate gamma was accepted in the u-component but not for the v-component.

10 Km. The u-component appears to be normal whereas the gamma is accepted in the v-component.

12 Km. Both distributions appear to be suspect for the u-component and the gamma is accepted for v. Normality is also rejected for v by considering the marginal distributions.

14 Km. Neither distribution is acceptable for u and the gamma is perhaps better for v.

6. CONCLUSIONS

The wind gust data was analyzed using some new procedures for the bivariate gamma distribution. The analysis was meant to be informative, in that it represents examples for some of the analytical procedure. The analysis is not meant to be completel - thorough. Hence, there are still some unresolved questions concerning the applicability of the bivariate gamma for modeling wind gust data. One suspects that neither the normal nor the gamma are completely appropriate, however, perhaps both could provide acceptable results for defining engineering constraints.

As mentioned in the paper the test for gamma is a Chi-square type procedure which has inherent problems which does not lend itself to easy data independent analysis. Instead it requires judicious selection of parameters. This analysis did not take advantage of this option, hence, the rejection of the gamma could be attributable to poor cell location choices.

Every data set was analyzed using a test for equality of shape parameters as proposed by Tubbs (1983). This hypothesis of equal shape parameters was rejected in every case.

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APPENDIX A

Univariate summary statistics generated using PROC UNIVARIATE is the Statistical Analysis System.

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The next four pages contain the summary statistics for each of the univariate variables Au, Lu, Av, and Lv, respectively. The reference altitude is

ALTITUDE = 4000

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VARIABLE=AU

MOMENTS

N	150	SUM WGTS	150
MEAN	1.39984	SUM	209.976
STD DEV	0.706416	VARIANCE	0.499023
SKEWNESS	0.682619	KURTOSIS	0.239346
USS	368.287	CSS	74.3545
CV	50.464	STD MEAN	0.0576786
T:MEAN=0	24.2697	PROB>|T|	0.0001
SGN RANK	5662.5	PROB>|S|	0.0001
NUM ^= 0	150		
D:NORMAL	0.0777153	PROB>D	0.024

QUANTILES(DEF=4)

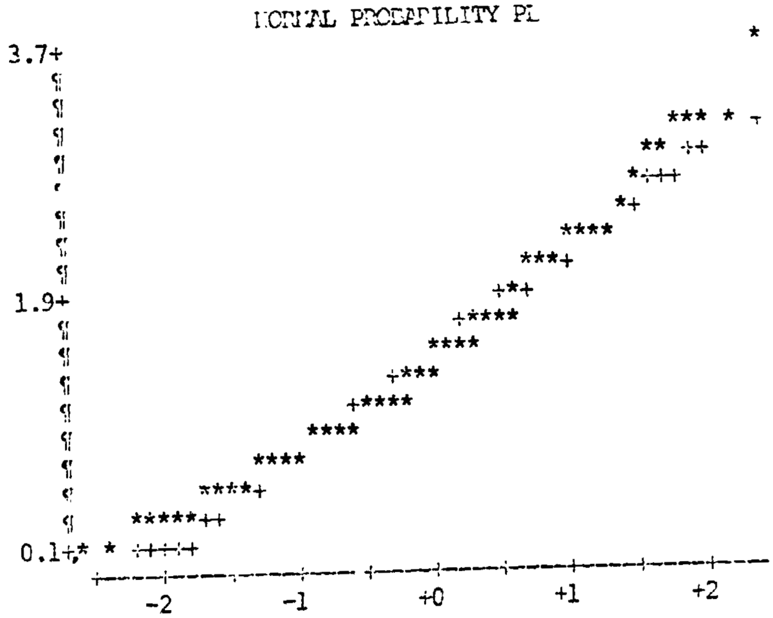
100% MAX	3.67	99%	3.40989
75% Q3	1.7825	95%	2.863
50% MED	1.295	90%	2.339
25% Q1	0.8775	10%	0.59
0% MIN	0.05	5%	0.3985
		1%	0.06836
RANGE	3.62		
Q3-Q1	0.905		
MODE	0.68		

EXTREMES

LOWEST	HIGHEST
0.05	3.12
0.086	3.13
0.28	3.13
0.3	3.16
0.33	3.67

```
STEM LEAF                      #  BOXPLOT
  36 7                         1     0
  34
  32
  30 2336                      4     0
  28 047                       3     |
  26 3                         1     |
  24 52                        2     |
  22 00178034556              11     |
  20 390135799                 9     |
  18 12318                     5     |
  16 14777913345889           14  +-----+
  14 12356678990035666889     20  |     |
  12 1112346725677            13  *--+--*
  10 123334567789055678       18  |     |
   8 1334567701112356889      19  +-----+
   6 23688891235699           14     |
   4 347367899                 9     |
   2 80356                     5     |
   0 59                        2     |
     ----+----+----+----+
 MULTIPLY STEM.LEAF BY 10**-01
```

NORMAL PROBABILITY PLOT



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VARIABLE=LU

MOMENTS

N	150	SUM WGTS	150
MEAN	808.968	SUM	121345
STD DEV	344.685	VARIANCE	118808
SKEWNESS	0.694665	KURTOSIS	0.642141
USS	115866718	CSS	17702335
CV	42.608	STD MEAN	28.1434
T:MEAN=0	28.7445	PROB>¦T¦	0.0001
SGN RANK	5662.5	PROB>¦S¦	0.0001
NUM ^= 0	150		
D:NORMAL	0.0689676	PROB>D	0.080

QUANTILES(DEF=4)

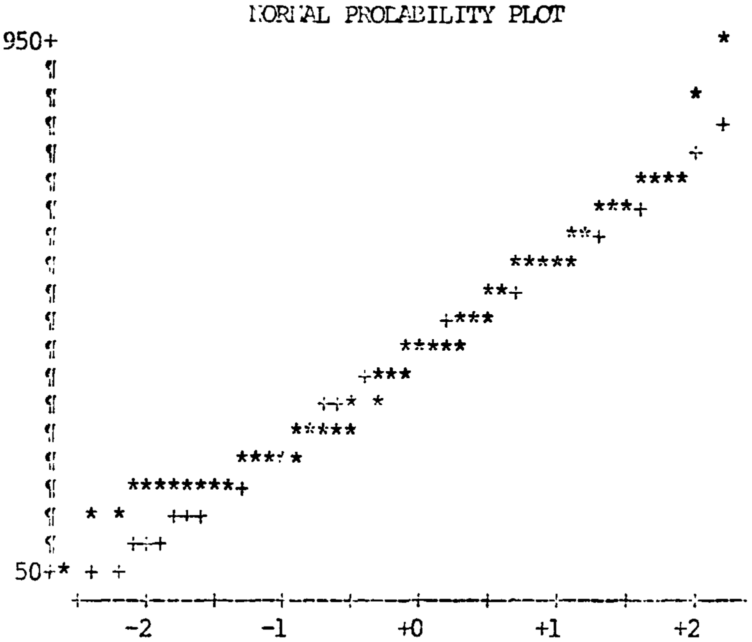
100% MAX	1974.9	99%	1964.6
75% Q3	1059	95%	1415.38
50% MED	786.45	90%	1277.49
25% Q1	537.3	10%	395.91
0% MIN	88.3	5%	347.845
		1%	158.629
RANGE	1886.6		
Q3-Q1	521.7		
MODE	702.4		

EXTREMES

LOWEST	HIGHEST
88.3	1439.1
226.2	1488.7
246.8	1792.7
304.7	1954.7
334.4	1974.9

```
STEM LEAF                     #  BOXPLOT
  19 57                       2     0
  18
  17 9                        1     ¦
  16                                ¦
  15                                ¦
  14 03449                    5     ¦
  13 12349                    5     ¦
  12 01789                    5     ¦
  11 01113334556679          14     ¦
  10 00123666778             11  +-----
   9 0011234 ·89             11  ¦
   8 0112233334556777899     19  ¦  +
   7 000335566778899         15  *-----*
   6 122245689999            12  ¦     ¦
   5 001223445566677777899   21  +-----+
   4 000122245677789         15     ¦
   3 03455567799             11     ¦
   2 35                       2     ¦
   1                                ¦
   0 9                        1     ¦
     ----+----+----+----+-
MULTIPLY STEM.LEAF BY 10**+02
```

NORMAL PROBABILITY PLOT



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VARIABLE=AV

MOMENTS

N	150	SUM WGTS	150	100% MAX	5.96	99%	4.93488
MEAN	1.4844	SUM	222.66	75% Q3	1.86	95%	3.012
STD DEV	0.822324	VARIANCE	0.676217	50% MED	1.405	90%	2.461
SKEWNESS	1.59111	KURTOSIS	5.05816	25% Q1	0.9	10%	0.673
USS	431.273	CSS	100.756	0% MIN	0.03	5%	0.51
CV	55.3977	STD MEAN	0.0671425			1%	0.03
T:MEAN=0	22.1082	PROB>|T|	0.0001	RANGE	5.93		
SGN RANK	5662.5	PROB>|S|	0.0001	Q3-Q1	0.96		
NUM ^= 0	150			MODE	1		
D:NORMAL	0.0905899	PROB>D	<0.01				

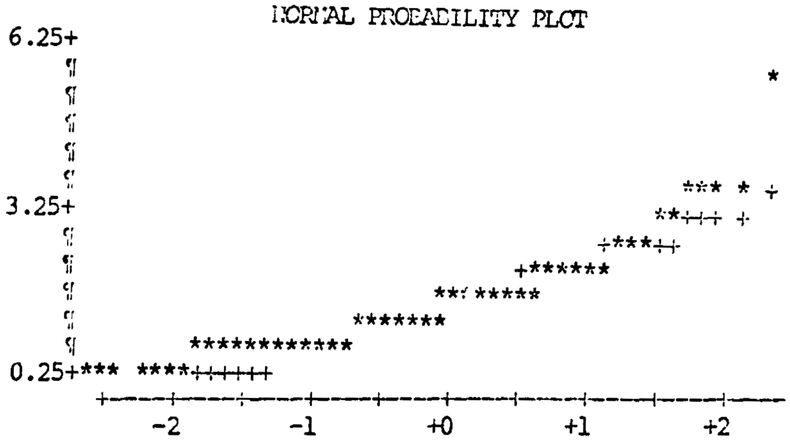
QUANTILES(DEF=4)

EXTREMES

LOWEST	HIGHEST
0.03	3.54
0.03	3.54
0.07	3.8
0.1	3.95
0.15	5.96

```
STEM LEAF                                          #  BOXPLOT
   6 0                                             1
   5                                                     *
   5
   4
   4
   3 5589                                          4     0
   3 11                                            2     |
   2 55556799                                      8     |
   2 00011111222222334                            17     |
   1 5555555566666666777777777888888888999999     39  +-----+
   1 000000000000111111111222222223333444444      39  *--+--*
   0 5556666677777888888888888899999999           34  +-----+
   0 001112                                        6     |
     ----+----+----+----+----+----+----+----
```

NORMAL PROBABILITY PLOT



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```
VARIABLE=LV

                    MOMENTS                                           QUANTILES(DEF=4)

N               150   SUM WGTS          150   100% MAX     2017.1     99%     1865.52
MEAN        723.427   SUM            108514    75% Q3     932.275     95%     1315.47
STD DEV     329.434   VARIANCE       108527    50% MED      667.1     90%     1152.19
SKEWNESS   0.789547   KURTOSIS      1.17774    25% Q1     500.975     10%      350.09
USS        94672440   CSS          16170518     0% MIN       96.1      5%      244.26
CV           45.538   STD MEAN      26.8982                            1%     104.566
T:MEAN=0     26.895   PROB>|T|       0.0001   RANGE          1921
SGN RANK     5662.5   PROB>|S|       0.0001   Q3-Q1         431.3
NUM ^= 0        150                           MODE          601.1
D:NORMAL  0.0776302   PROB>D          0.024

              EXTREMES

    LOWEST            HIGHEST
      96.1             1427.3
     112.7             1477.5
     122.7             1490.1
       134             1719.9
     167.1             2017.1

STEM LEAF                                # BOXPLOT
  20 2                                   1     0
  19
  18
  17 2                                   1     0
  16
  15
  14 1389                                4     |
  13 6                                   1     |
  12 567788                              6     |
  11 45559                               5     |
  10 0237779                             7     |
   9 001223344555668888                 18  +-----+
   8 011344566789                       12  |     |
   7 002222334555789                    15  |  +  |          NORMAL PROBABILITY PLOT
   6 000001123333445667777999           24  *-----*
   5 00222233344446677888               20  +-----+
   4 11222333334446799                  17     |
   3 1223555688                         10     |
   2 278                                 3     |
   1 012377                              6     |
   0
     ----+----+----+----+----
     MULTIPLY STEM.LEAF BY 10**+02
```

```
      |                                                              *
      |
      |                                                            *
      |                                                              +
      |                                                        **** +
      |                                                       *+++
1250+ |                                                    ***+
      |                                                  **+
1050+ |                                               +***
      |                                            *****
 850+ |                                         ****
      |                                      +***
 650+ |                                   *****
      |                               *****
 450+ |                           ****+
      |                       ****++
 250+ |                     **+++
      |    *  ****++
  50+*   +  +++
      +----+----+----+----+----+----+----+----+----+----+
              -2        -1        +0        +1        +2
```

The next four pages contain the summary statistics for each of the univariate variables Au, Lu, Av, and Lv, respectively. The reference altitude is

ALTITUDE = 6000

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VARIABLE=AU

MOMENTS

N	150	SUM WGTS	150
MEAN	1.34487	SUM	201.73
STD DEV	0.735961	VARIANCE	0.541639
SKEWNESS	0.630477	KURTOSIS	0.966481
USS	352.004	CSS	80.7041
CV	54.7237	STD MEAN	0.060091
T:MEAN=0	22.3805	PROB>|T|	0.0001
SGN RANK	5662.5	PROB>|S|	0.0001
NUM ^= 0	150		
D:NORMAL	0.0470466	PROB>D	>0.15

QUANTILES(DEF=4)

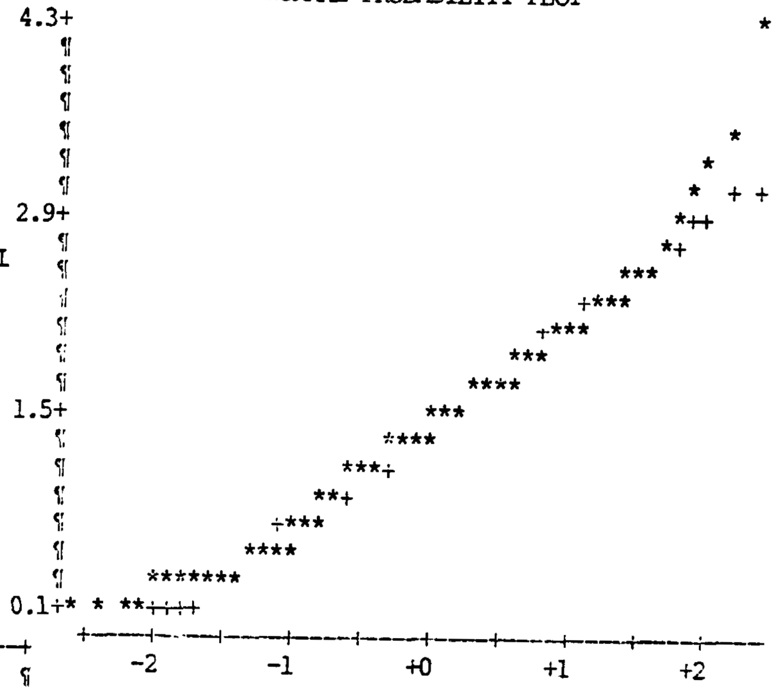
100% MAX	4.22	99%	3.83239
75% Q3	1.785	95%	2.5415
50% MED	1.33	90%	2.209
25% Q1	0.8	10%	0.36
0% MIN	0.03	5%	0.2655
		1%	0.0656999
RANGE	4.19		
Q3-Q1	0.985		
MODE	1.67		

EXTREMES

LOWEST	HIGHEST
0.03	2.83
0.1	3.08
0.15	3.26
0.17	3.46
0.2	4.22

```
STEM LEAF                         #  BOXPLOT
  42 2                            1     0
  40
  38
  36
  34 6                            1     0
  32 6                            1     |
  30 8                            1     |
  28 3                            1     |
  26 8                            1     |
  24 1118                         4     |
  22 012901                       6     |
  20 1244483348                  10     |
  18 02340157778                 11     |
  16 0011477777899902238         19  +-----+
  14 234556677844                12  |     |
  12 1223344450222333555599      22  *--+--*
  10 134466667145                12  |     |
   8 001224591588                12  +-----+
   6 33670055                     8     |
   4 012790335668                12     |
   2 016770124666                12     |
   0 3057                         4     |
     ----+----+----+----+
MULTIPLY STEM.LEAF BY 10**-01
```

NORMAL PROBABILITY PLOT



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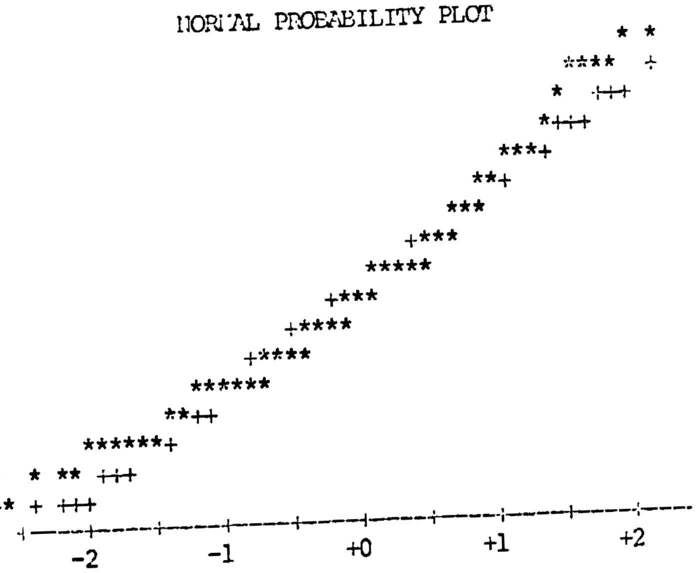
```
VARIABLE=LU

              MOMENTS                                                 QUANTILES(DEF=4)

N                150   SUM WGTS          150   100% MAX    1646.5      99%     1635.59
MEAN         748.856   SUM            112328    75% Q3    940.675      95%     1470.54
STD DEV      351.218   VARIANCE       123354    50% MED     711.3      90%     1262.96
SKEWNESS    0.552222   KURTOSIS    -0.132913    25% Q1    482.975      10%      323.72
USS        102497549   CSS          18379753     0% MIN      65.9       5%      225.55
CV           46.9006   STD MEAN      28.6768                            1%      104.15
T:MEAN=0     26.1136   PROB>|T|       0.0001   RANGE       1580.6
SGN RANK      5662.5   PROB>|S|       0.0001   Q3-Q1        457.7
NUM ^= 0         150                           MODE         634.3
D:NORMAL   0.0812055   PROB>D          0.016

            EXTREMES

     LOWEST          HIGHEST
       65.9           1560.4
      140.9           1582.9
      151.8           1607.6
      194.8           1625.1
      205.2           1646.5
```

```
STEM LEAF                   #  BOXPLOT
  16 135                    3     0      1650+
  15 5568                   4     |          |
  14 1                      1     |          |
  13 0177                   4     |          |
  12 0003799                7     |          |
  11 34568                  5     |          |
  10 112244557              9     |          |
   9 0000112345778         13  +-----+       |
   8 0123334555667788899   18  |     |    850+
   7 0124667788999         13  *--+--*       |
   6 01133344445579        14  |     |       |
   5 0111222234445556789   19  |     |       |
   4 1122445556666788899   19  +-----+       |
   3 0012234799            10     |          |
   2 1123688                7     |          |
   1 459                    3     |          |
   0 7                      1     |        50+
     ----+----+----+----+
MULTIPLY STEM.LEAF BY 10**+02
```

NORMAL PROBABILITY PLOT



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VARIABLE=AV

```
                 MOMENTS                                              QUANTILES(DEF=4)

N                150   SUM WGTS          150   100% MAX       4.16       99%        3.9254
MEAN         1.55953   SUM            233.93    75% Q3        2.07       95%        3.0305
STD DEV     0.788246   VARIANCE     0.621332    50% MED      1.495       90%         2.632
SKEWNESS    0.565903   KURTOSIS     0.305502    25% Q1      0.9675       10%         0.612
USS            457.4   CSS           92.5785     0% MIN       0.07        5%         0.334
CV           50.5437   STD MEAN      0.06436                             1%        0.1057
T:MEAN=0     24.2314   PROB½¶T¶       0.0001     RANGE        4.09
SGN RANK      5662.5   PROB½¶S¶       0.0001     Q3-Q1      1.1025
NUM ^= 0         150                             MODE         0.71
D:NORMAL    0.054269   PROB½D          ½0.15

             EXTREMES

       LOWEST          HIGHEST
         0.07             3.32
         0.14             3.37
         0.19              3.6
          0.2              3.7
         0.21             4.16

STEM LEAF                       #  BOXPLOT                         NORMAL PROBABILITY PLOT
  40 6                          1     0
  38                                        4.1+
  36 00                         2     ¶          ¶
  34                                  ¶     3.7+                                                    *  *
  32 127                        3     ¶          ¶
  30 8                          1     ¶     3.3+                                                  ***+ +
  28 69                         2     ¶          ¶                                                * ++
  26 447116                     6     ¶     2.9+                                                *++
  24 23446756                   8     ¶          ¶                                            ***
  22 47059                      5     ¶     2.5+                                          ***+
  20 013445001234566           15  +-----+       ¶                                          *+
  18 1467859                    7  ¶     ¶  2.1+                                        ****
  16 02335579991358999         17  ¶     ¶       ¶                                     ***
  14 122233348811122479        18  *--+--*  1.7+                                    +***
  12 2456899146789             13  ¶     ¶       ¶                               ****
  10 25669013467               11  ¶     ¶  1.3+                               ***
   8 00113589013467788         17  +-----+       ¶                          +***
   6 1356111267                10     ¶     0.9+                        *****
   4 87899                      5     ¶          ¶                   ***+
   2 017979                     6     ¶     0.5+                **+
   0 749                        3     ¶          ¶          *****+
     ----+----+----+----+                   0.1+*  *  *+++
   MULTIPLY STEM.LEAF BY 10**-01               +----+----+----+----+----+----+----+----+----+----+
                                                   -2        -1        +0        +1        +2
```

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VARIABLE=LV

MOMENTS

N	150	SUM WGTS	150
MEAN	833.661	SUM	125049
STD DEV	359.396	VARIANCE	129165
SKEWNESS	0.614472	KURTOSIS	0.699756
USS	123494168	CSS	19245651
CV	43.1106	STD MEAN	29.3446
T:MEAN=0	28.4094	PROB>¦T¦	0.0001
SGN RANK	5662.5	PROB>¦S¦	0.0001
NUM ^= 0	150		
D:NORMAL	0.0717385	PROB>D	0.057

QUANTILES(DEF=4)

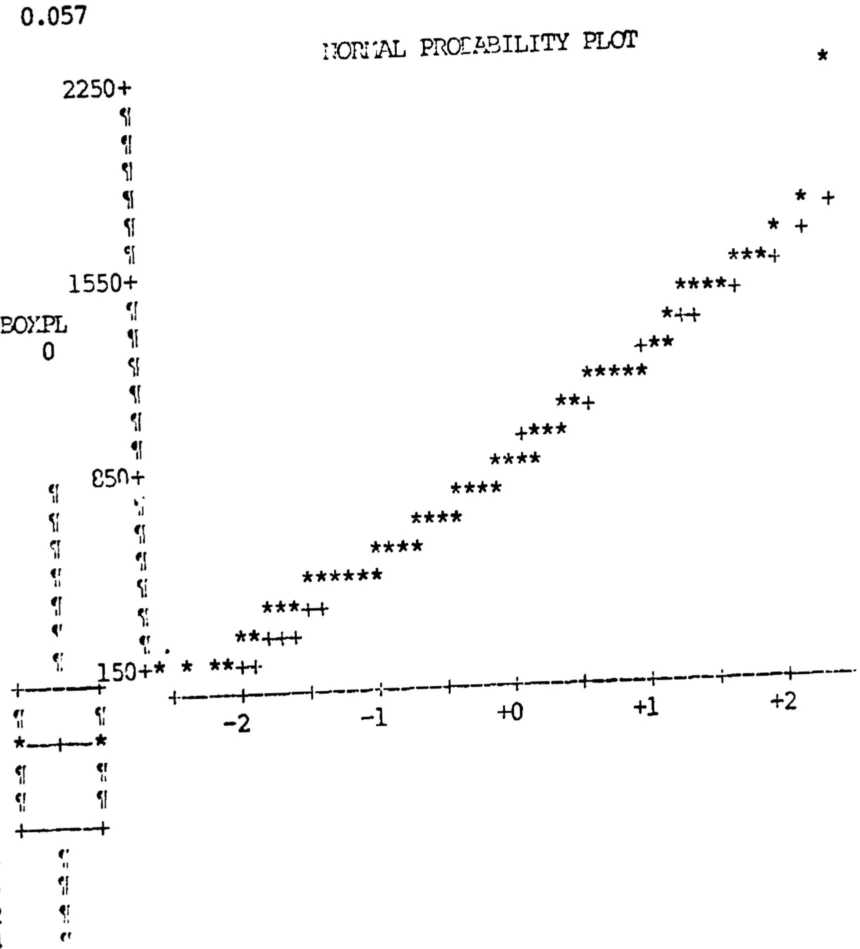
100% MAX	2224.4	99%	1959.25
75% Q3	1095.12	95%	1491.33
50% MED	805.7	90%	1307.57
25% Q1	588.275	10%	411.82
0% MIN	116.4	5%	318.755
		1%	135.015
RANGE	2108		
Q3-Q1	506.85		
MODE	642.3		

EXTREMES

LOWEST	HIGHEST
116.4	1559.2
152.9	1591.4
170	1677.7
193.8	1704.5
211.8	2224.4

STEM	LEAF	#
22	2	1
21		
20		
19		
18		
17	0	1
16	8	1
15	169	3
14	116799	6
13	127	3
12	003389	6
11	1122234455567889	16
10	00124445689	11
9	02334566	8
8	001122223444566677789	21
7	11223356688999	14
6	0112333344444555668	19
5	0224447889999	13
4	0011144455556788	16
3	13366	5
2	16	2
1	2579	4

MULTIPLY STEM.LEAF BY 10**+02



The next four pages contain the summary statistics for each of the univariate variables Au, Lu, Av, and Lv, respectively. The reference altitude is

ALTITUDE = 8000

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VARIABLE=AU

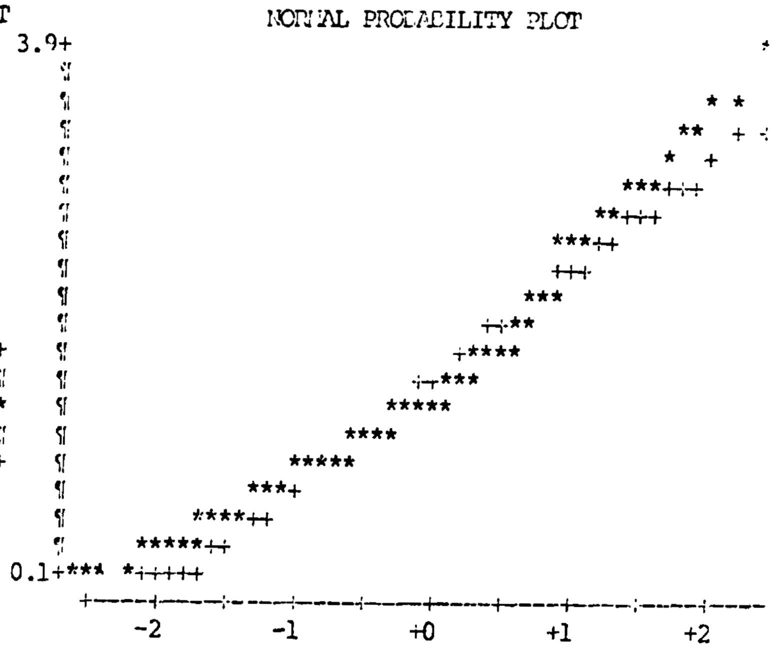
MOMENTS				QUANTILES (DEF=4)			
N	150	SUM WGTS	150	100% MAX	3.87	99%	3.6507
MEAN	1.43033	SUM	214.55	75% Q3	1.76	95%	2.9645
STD DEV	0.780668	VARIANCE	0.609442	50% MED	1.295	90%	2.627
SKEWNESS	0.804124	KURTOSIS	0.252035	25% Q1	0.87	10%	0.57
USS	397.685	CSS	90.8069	0% MIN	0.09	5%	0.3855
CV	54.5794	STD MEAN	0.0637413			1%	0.09
T:MEAN=0	22.4397	PROB>|T|	0.0001	RANGE	3.78		
SGN RANK	5662.5	PROB>|S|	0.0001	Q3-Q1	0.89		
NUM ^= 0	150			MODE	0.8		
D:NORMAL	0.10402	PROB>D	<0.01				

EXTREMES

LOWEST	HIGHEST
0.09	3.2
0.09	3.27
0.12	3.41
0.25	3.44
0.32	3.87

```
STEM LEAF                     #  BOXPLOT
  38 7                        1     0
  36
  34 14                       2     0
  32 07                       2     0
  30 7                        1     0
  28 32367                    5     |
  26 03578                    5     |
  24 24489017                 8     |
  22 8                        1     |
  20 116348                   6     |
  18 18012                    5     |
  16 112222401125559         15  +-----+
  14 011334679933779         15  |  +  |
  12 01224556779001125688    20  *-----*
  10 011222347891223457      18  |     |
   8 00000227779013488       17  +-----+
   6 14502335688             11     |
   4 0136667779              10     |
   2 52889                    5     |
   0 992                      3     |
     ----+----+----+----+
MULTIPLY STEM.LEAF BY 10**-01
```

NORMAL PROBABILITY PLOT



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VARIABLE=LU

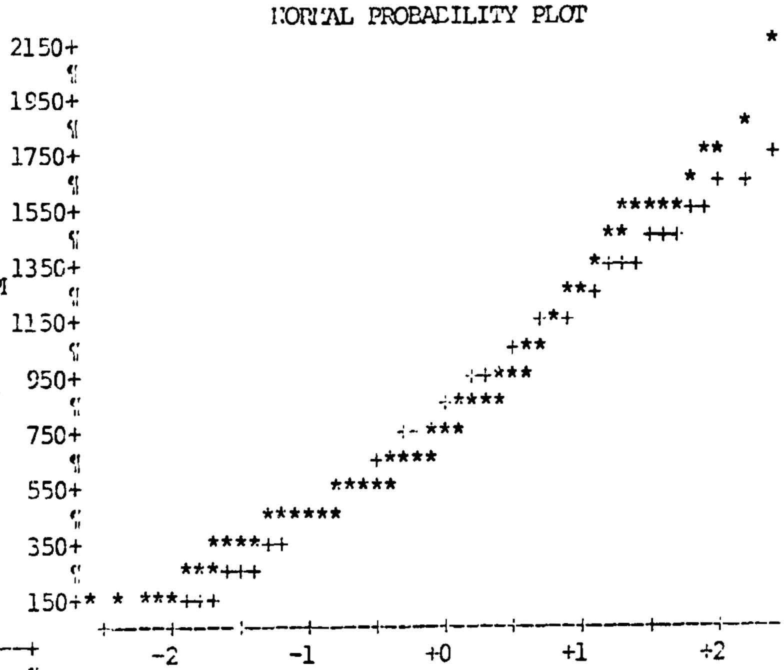
MOMENTS				QUANTILES(DEF=4)			
N	150	SUM WGTS	150	100% MAX	2150.7	99%	1972.5
MEAN	792.487	SUM	118873	75% Q3	975.725	95%	1566.55
STD DEV	391.179	VARIANCE	153021	50% MED	703.95	90%	1374.45
SKEWNESS	0.895196	KURTOSIS	0.522044	25% Q1	503.075	10%	391.84
USS	117005567	CSS	22800141	0% MIN	114.8	5%	282.655
CV	49.3609	STD MEAN	31.9396			1%	147.032
T:MEAN=0	24.812	PROB>¦T¦	0.0001	RANGE	2035.9		
SGN RANK	5662.5	PROB>¦S¦	0.0001	Q3-Q1	472.65		
NUM ^= 0	150			MODE	114.8		
D:NORMAL	0.108007	PROB>D	<0.01				

EXTREMES

LOWEST	HIGHEST
114.8	1686.1
178	1717.8
186	1798.1
186.8	1801.3
196.6	2150.7

```
STEM LEAF                       #  BOXPLOT
  21 5                          1     0
  20
  19
  18 00                         2     0
  17 2                          1     0
  16 9                          1     0
  15 0223677                    7     ¦
  14 78                         2     ¦
  13 2558                       4     ¦
  12 1477889                    7     ¦
  11 002777                     6     ¦
  10 22235                      5     ¦
   9 000122345678              12  +-----+
   8 134444555666779           15  ¦     ¦
   7 000112223445699           15  *--+--*
   6 011223445566677899        18  ¦     ¦
   5 000001223444456889999     21  +-----+
   4 00011122334556669         17     ¦
   3 03345899                   8     ¦
   2 0779                       4     ¦
   1 1899                       4     ¦
     ----+----+----+----+
 MULTIPLY STEM.LEAF BY 10**+02
```

NORMAL PROBABILITY PLOT



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VARIABLE=AV

MOMENTS

N	150	SUM WGTS	150
MEAN	1.61353	SUM	242.03
STD DEV	0.872608	VARIANCE	0.761444
SKEWNESS	0.558956	KURTOSIS	-0.351765
USS	503.979	CSS	113.455
CV	54.0806	STD MEAN	0.0712481
T:MEAN=0	22.6467	PROB>|T|	0.0001
SGN RANK	5662.5	PROB>|S|	0.0001
NUM ^= 0	150		
D:NORMAL	0.0807145	PROB>D	0.017

QUANTILES(DEF=4)

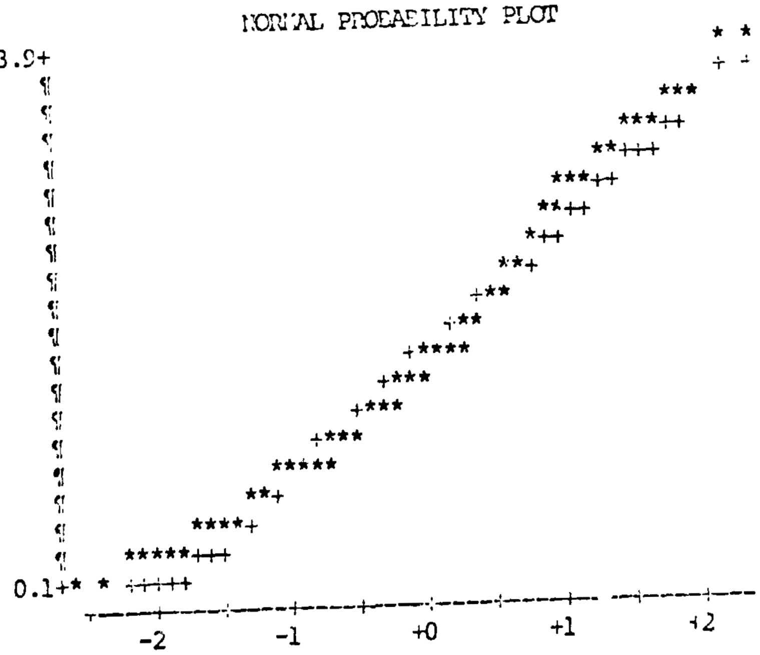
100% MAX	3.95	99%	3.9092
75% Q3	2.2225	95%	3.2645
50% MED	1.48	90%	2.919
25% Q1	0.96	10%	0.531
0% MIN	0.1	5%	0.3355
		1%	0.1153
RANGE	3.85		
Q3-Q1	1.2625		
MODE	1.04		

EXTREMES

LOWEST	HIGHEST
0.1	3.43
0.13	3.48
0.23	3.53
0.29	3.87
0.29	3.95

```
STEM LEAF                   #  BOXPLOT
  38 75                     2     |
  36                              |
  34 383                    3     |
  32 3677                   4     |
  30 688                    3     |
  28 124771267              9     |
  26 244                    3     |
  24 090458                 6     |
  22 23678239               8  +-----+
  20 000024                 6  |     |
  18 27789559               8  |     |
  16 2233889001344599      16  |  +  |
  14 145560125679          12  *-----*
  12 0112555677013359      16  |     |
  10 0244457802569         13  |     |
   8 0677891144566779      16  +-----+
   6 157003479              9     |
   4 02479134               8     |
   2 399034                 6     |
   0 03                     2     |
     ----+----+----+----+
MULTIPLY STEM.LEAF BY 10**-01
```

NORMAL PROBABILITY PLOT



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VARIABLE=LV

MOMENTS

N	150	SUM WGTS	150
MEAN	842.257	SUM	126339
STD DEV	398.748	VARIANCE	159000
SKEWNESS	1.12027	KURTOSIS	1.65767
USS	130100556	CSS	23690944
CV	47.3427	STD MEAN	32.5576
T:MEAN=0	25.8698	PROB>|T|	0.0001
SGN RANK	5662.5	PROB>|S|	0.0001
NUM ^= 0	150		
D:NORMAL	0.107964	PROB>D	<0.01

QUANTILES(DEF=4)

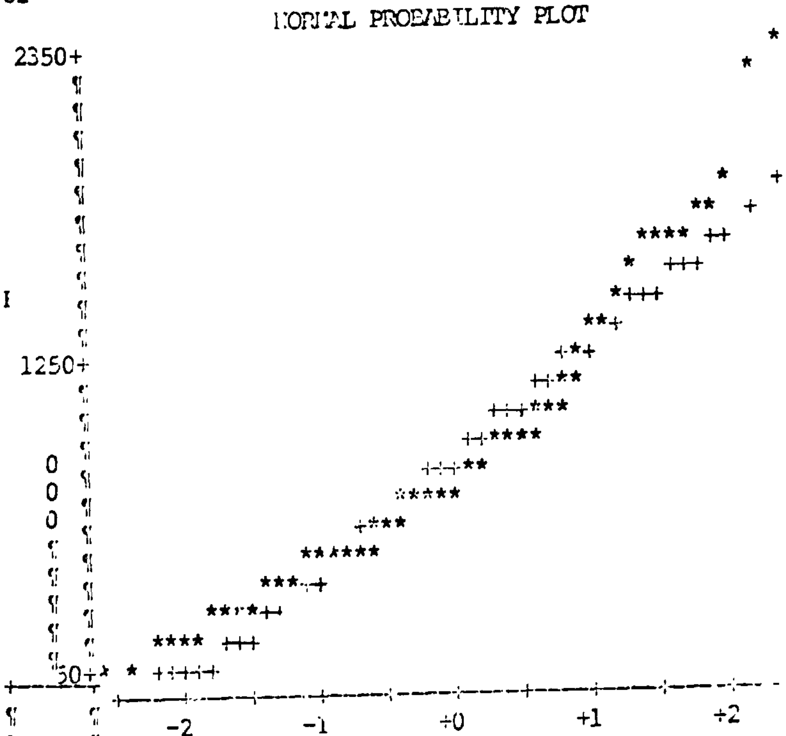
100% MAX	2314.7	99%	2299.3
75% Q3	1001.9	95%	1654.59
50% MED	773.8	90%	1412.32
25% Q1	580.875	10%	423.16
0% MIN	115.3	5%	328.81
		1%	123.001
RANGE	2199.4		
Q3-Q1	421.025		
MODE	115.3		

EXTREMES

LOWEST	HIGHEST
115.3	1746.5
130.4	1794.7
235	1839
235.9	2284.5
278.1	2314.7

```
STEM LEAF                           #   BOXPLOT
  23 1                              1
  22 8                              1
  21
  20
  19
  18 4                              1     0
  17 59                             2     0
  16 123479                         6     0
  15 06                             2     |
  14 016                            3     |
  13 2369                           4     |
  12 1459                           4     |
  11 039                            3     |
  10 00023456778 9                 12  +-----+
   9 122234556677799               15  |     |
   8 112456677789                  12  |  +  |
   7 01122333334567777788999999     26  *-----*
   6 000022333456899               15  |     |
   5 00011222355788899999           20  +-----+
   4 0234578899                    10     |
   3 03344557                       8     |
   2 348                            3     |
   1 23                             2     |
     ----+----+----+----+----+
MULTIPLY STEM.LEAF BY 10**+02
```

NORMAL PROBABILITY PLOT



The next four pages contain the summary statistics for each of the univariate variables Au, Lu, Av, and Lv, respectively. The reference altitude is

ALTITUDE = 10,000

C-2

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VARIABLE=AU

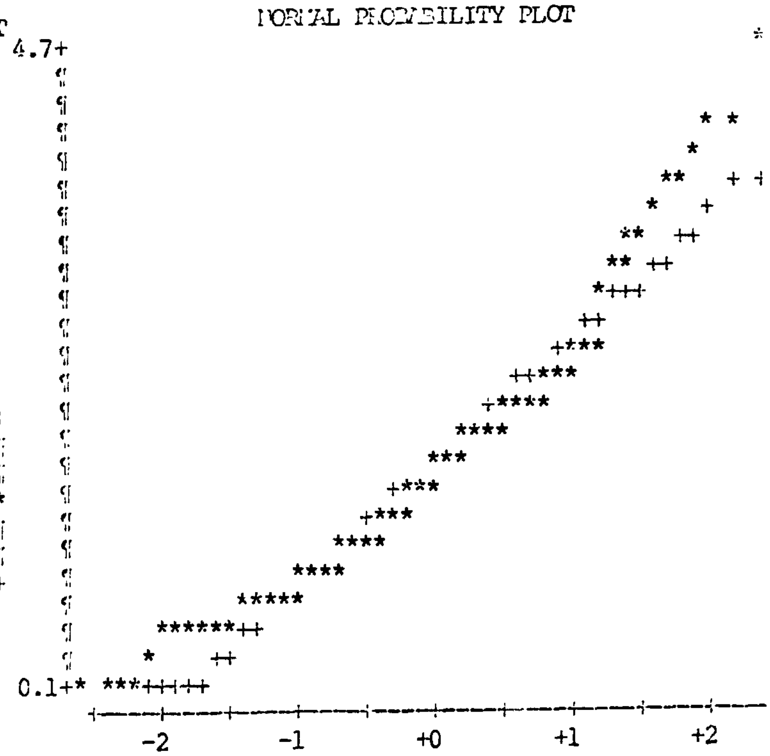
MOMENTS				QUANTILES(DEF=4)			
N	150	SUM WGTS	150	100% MAX	4.73	99%	4.42909
MEAN	1.59273	SUM	238.91	75% Q3	2.0825	95%	3.414
STD DEV	0.879401	VARIANCE	0.773346	50% MED	1.45	90%	2.756
SKEWNESS	0.923389	KURTOSIS	1.03196	25% Q1	0.9025	10%	0.591
USS	495.748	CSS	115.229	0% MIN	0.09	5%	0.44
CV	55.2133	STD MEAN	0.0718028			1%	0.1308
T:MEAN=0	22.1821	PROB>¦T¦	0.0001	RANGE	4.64		
SGN RANK	5662.5	PROB>¦S¦	0.0001	Q3-Q1	1.18		
NUM ^= 0	150			MODE	1.33		
D:NORMAL	0.0733202	PROB>D	0.046				

EXTREMES

LOWEST	HIGHEST
0.09	3.71
0.17	3.96
0.17	4
0.38	4.14
0.4	4.73

```
STEM LEAF               #  BOXPLOT
  46 3                  1     0
  44
  42
  40 04                 2     0
  38 6                  1     0
  36 61                 2     |
  34 9                  1     |
  32 037                3     |
  30 67                 2     |
  28 49                 2     |
  26 8                  1     |
  24 34134              5     |
  22 01370177           8     |
  20 122568903444679   15  +-----+
  18 1368889123458     13  |     |
  16 22679024467889    14  |     |
  14 01444556355       11  *--+--*
  12 1456793333356     13  |     |
  10 66780122233599    14  |     |
   8 1233568816677     13  +-----+
   6 01234666690227    14     |
   4 03445755779       11     |
   2 8                  1     |
   0 977                3     |
     ----+----+----+----+
MULTIPLY STEM.LEAF BY 10**-01
```

NORMAL PROBABILITY PLOT



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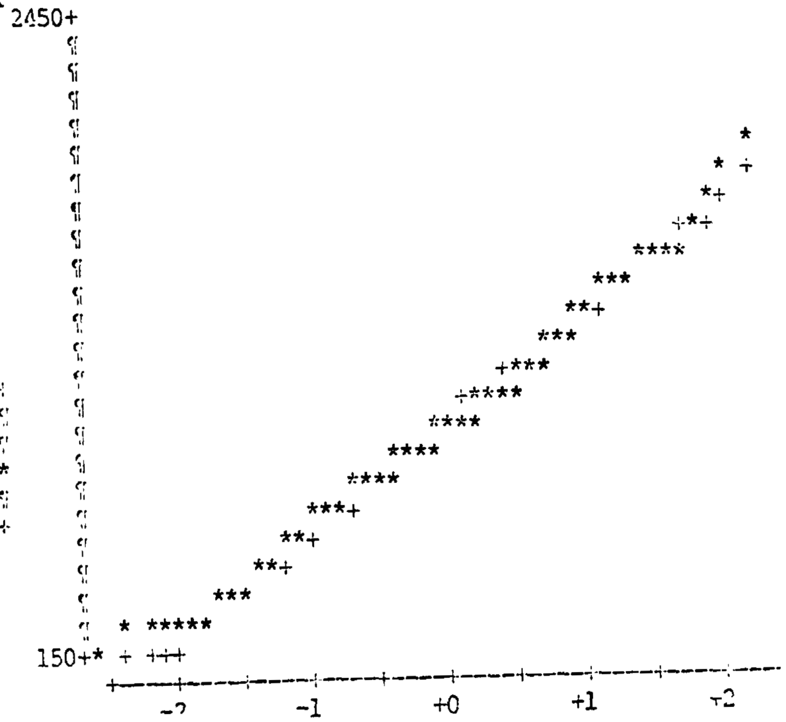
VARIABLE=LU

```
                 MOMENTS                                         QUANTILES(DEF=4)

N              150   SUM WGTS          150   100% MAX      2466      99%      2203.29
MEAN       922.983   SUM            138447    75% Q3    1175.47      95%      1549.76
STD DEV    385.088   VARIANCE       148293    50% MED    899.25      90%       1463.3
SKEWNESS  0.581313   KURTOSIS     0.982827    25% Q1    671.925      10%       418.73
USS      149880139   CSS          22095588     0% MIN     173.5       5%       299.57
CV         41.7221   STD MEAN      31.4423                            1%      196.858
T:MEAN=0   29.3548   PROB>|T|       0.0001   RANGE       2292.5
SGN RANK    5662.5   PROB>|S|       0.0001   Q3-Q1       503.55
NUM ^= 0       150                           MODE         854.2
D:NORMAL 0.0627019   PROB>D          >0.15

          EXTREMES

     LOWEST          HIGHEST
      173.5           1665.4
      219.3           1737.3
      224.7           1817.6
      250.3           1950.9
      276.7             2466
```

```
STEM LEAF                     #  BOXPLOT
  24 7                        1     0
  23
  22
  21
  20
  19 5                        1     0
  18 2                        1     |
  17 4                        1     |
  16 7                        1     |
  15 004557                   6     |
  14 0266788                  7     |
  13 136788                   6     |
  12 023445679                9     |
  11 023367889 9             10  +-----+
  10 000112224556789         15  |     |
   9 000122334455557899      18  |  +  |
   8 023455666667889 9       16  *-----*
   7 011223445678 8999       16  |     |
   6 1134666778889           13  +-----+
   5 23446799                 8     |
   4 12445559                 8     |
   3 0017899                  7     |
   2 22589                    5     |
   1 7                        1     |
     ----+----+----+----+
MULTIPLY STEM.LEAF BY 10**+02
```

NORMAL PROBABILITY PLOT



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VARIABLE=AV

MOMENTS

N	150	SUM WGTS	150
MEAN	1.9398	SUM	290.97
STD DEV	1.13858	VARIANCE	1.29637
SKEWNESS	0.877054	KURTOSIS	0.909461
USS	757.583	CSS	193.159
CV	58.6959	STD MEAN	0.0929649
T:MEAN=0	20.8659	PROB>|T|	0.0001
SGN RANK	5662.5	PROB>|S|	0.0001
NUM ^= 0	150		
D:NORMAL	0.0739582	PROB>D	0.043

QUANTILES(DEF=4)

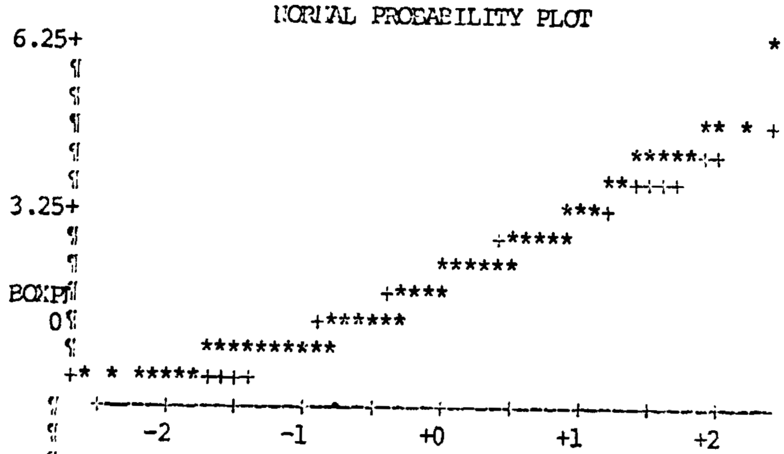
100% MAX	6.37	99%	5.59478
75% Q3	2.5725	95%	4.284
50% MED	1.825	90%	3.349
25% Q1	1.015	10%	0.691
0% MIN	0.07	5%	0.4665
		1%	0.0903999
RANGE	6.3		
Q3-Q1	1.5575		
MODE	0.89		

EXTREMES

LOWEST	HIGHEST
0.07	4.44
0.11	4.59
0.15	4.6
0.22	4.85
0.22	6.37

```
STEM LEAF                                  #  BOXPLOT
  6 4                                      1     0
  5
  5
  4 668                                    3     |
  4 0122344                                7     |
  3 559                                    3     |
  3 000112233333                          12     |
  2 5555666667788899                      17  +-----+
  2 000001111111222233333333              24  |     |
  1 555555667777888889999999              23  *--+--*
  1 00000111111112233334444444444         27  +-----+
  0 55666777777788888889999999            26     |
  0 1112234                                7     |
    ----+----+----+----+----+--
```

NORMAL PROBABILITY PLOT



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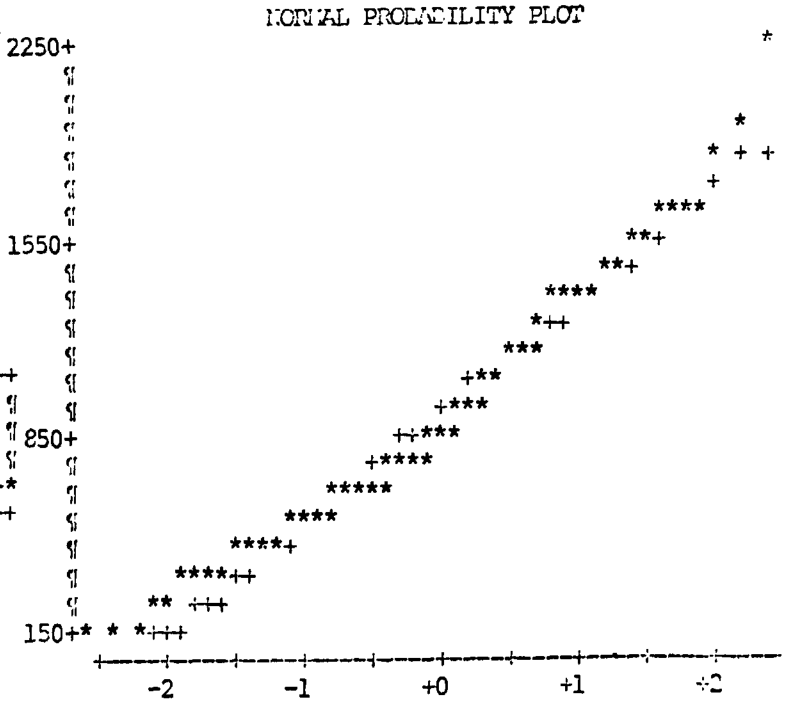
VARIABLE=LV

```
                  MOMENTS                                          QUANTILES(DEF=4)

N               150   SUM WGTS        150   100% MAX    2221.7     99%      2091.19
MEAN        886.825   SUM          133024    75% Q3    1172.22     95%      1596.77
STD DEV     399.875   VARIANCE     159900    50% MED    799.65     90%      1427.67
SKEWNESS    0.58784   KURTOSIS   0.042399    25% Q1    601.825     10%       409.93
USS       141794019   CSS        23825143     0% MIN     106.1      5%      338.805
CV          45.0907   STD MEAN    32.6497                            1%      127.775
T:MEAN=0    27.1618   PROB>|T|     0.0001   RANGE       2115.6
SGN RANK     5662.5   PROB>|S|     0.0001   Q3-Q1        570.4
NUM ^= 0        150                         MODE         959.5
D:NORMAL   0.109902   PROB>D        <0.01

             EXTREMES

   LOWEST          HIGHEST
    106.1           1663.3
    148.6           1695.8
    171.9           1821.4
    223.1           1965.8
    250.7           2221.7

STEM LEAF                    #  BOXPLOT
  22 2                       1     0
  21
  20
  19 7                       1     |
  18 2                       1     |
  17 0                       1     |
  16 166                     3     |
  15 2678                    4     |
  14 33678                   5     |
  13 012555777889           12     |
  12 046799                  6     |
  11 00134477889            11  +-----+
  10 1245669                 7  |     |
   9 011456666799           12  |     |
   8 0000111567779          13  |  +  |
   7 0000011222445778899    19  *-----*
   6 0000111223366677788    19  +-----+
   5 45555566778            11     |
   4 00001244555777         14     |
   3 23579                   5     |
   2 25                      2     |
   1 157                     3     |
     ----+----+----+----+
MULTIPLY STEM.LEAF BY 10**+02
```

NORMAL PROBABILITY PLOT



The next four pages contain the summary statistics for each of the univariate variables Au, Lu, Av, and Lv, respectively. The reference altitude is

ALTITUDE = 12,000

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```
VARIABLE=AU

              MOMENTS                                              QUANTILES(DEF=4)

N              150   SUM WGTS         150   100% MAX       6.7      99%      6.6796
MEAN       2.22487   SUM           333.73    75% Q3     2.8625      95%      5.2585
STD DEV    1.45363   VARIANCE     2.11305    50% MED     1.885      90%        4.29
SKEWNESS   1.08163   KURTOSIS    0.915241    25% Q1     1.1075      10%       0.735
USS        1057.35   CSS          314.845     0% MIN      0.06       5%       0.382
CV         65.3358   STD MEAN    0.118689                            1%      0.0651
T:MEAN=0   18.7454   PROB>|T|      0.0001   RANGE         6.64
SGN RANK    5662.5   PROB>|S|      0.0001   Q3-Q1        1.755
NUM ^= 0       150                          MODE          0.97
D:NORMAL  0.116829   PROB>D         <0.01

         EXTREMES

    LOWEST        HIGHEST
      0.06           6.29
      0.07           6.31
      0.16           6.34
       0.2           6.66
      0.23            6.7

STEM LEAF                                  #  BOXPLOT
   6 77                                    2     0
   6 333                                   3     0
   5
   5 2234                                  4     |
   4 56789                                 5     |
   4 01223                                 5     |
   3 556678899                             9     |
   3 001122223                             9     |
   2 5555555666777788                     16  +-----+
   2 00000122222233444444                 19  |  +  |
   1 555556667777777788888899999          26  *-----*
   1 00000000001111111222222233334        29  +-----+
   0 556666789999999                      15     |
   0 11222344                              8     |
     ----+----+----+----+----+----

                                          NORMAL PROBABILITY PLOT
                 6.75+                                                        * *
                      |                                                   ***
                      |                                                       + +
                      |                                               *** ++
                      |                                             **++++
                      |                                           **++
                      |                                        ***+
                      |                                      +***
                      |                                   ++****
                      |                                ++****
                      |                             +*****
                      |                         ******
                      |                 ********
                 0.25+* * *****+++++
                       +----+----+----+----+----+----+----+----+----+----+
                           -2        -1        +0        +1        +2
```

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VARIABLE=LU

MOMENTS			
N	150	SUM WGTS	150
MEAN	837.755	SUM	125663
STD DEV	408.852	VARIANCE	167160
SKEWNESS	1.05082	KURTOSIS	1.90283
USS	130181790	CSS	24906858
CV	48.8033	STD MEAN	33.3826
T:MEAN=0	25.0955	PROB>|T|	0.0001
SGN RANK	5662.5	PROB>|S|	0.0001
NUM ^= 0	150		
D:NORMAL	0.112428	PROB>D	<0.01

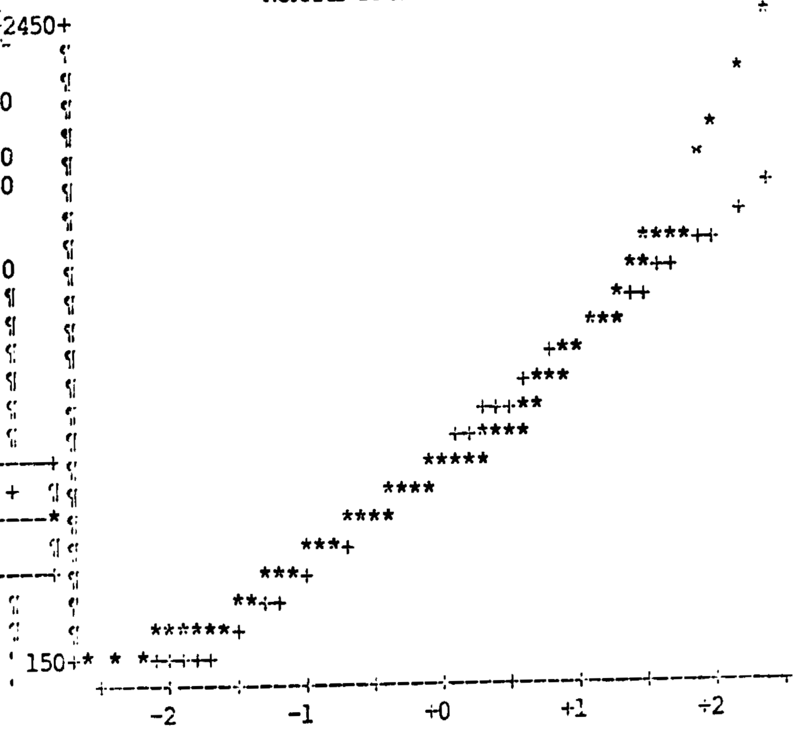
QUANTILES(DEF=4)			
100% MAX	2445	99%	2351.52
75% Q3	989.15	95%	1634.44
50% MED	793.95	90%	1352.38
25% Q1	556.375	10%	326.89
0% MIN	130.7	5%	258.33
		1%	138.707
RANGE	2314.3		
Q3-Q1	432.775		
MODE	809.1		

EXTREMES

LOWEST	HIGHEST
130.7	1695.9
146.4	1900.8
153	2005
208.7	2261.7
212.4	2445

```
STEM LEAF                              #  BOXPLOT
  24 4                                 1    *
  23
  22 6                                 1    0
  21
  20 0                                 1    0
  19 0                                 1    0
  18
  17 0                                 1    0
  16 168                               3    0
  15 089                               3    |
  14 3                                 1    |
  13 00134567                          8    |
  12 16669                             5    |
  11 1156679                           7    |
  10 04678                             5    |
   9 02223456667888 9                 15 +-----+
   8 00111222344455567788899          23 |  +  |
   7 0122333444445666799              19 *-----*
   6 0000135556677899                 16 |     |
   5 00001234555688                   14 +-----+
   4 03667789                          8    |
   3 1223488                           7    |
   2 11457789                          8    |
   1 355                               3    |
     ----+----+----+----+---
```

NORMAL PROBABILITY PLOT



VARIABLE=AV

MOMENTS

N	150	SUM WGTS	150
MEAN	2.4854	SUM	372.81
STD DEV	1.39227	VARIANCE	1.93841
SKEWNESS	0.600256	KURTOSIS	-0.185831
USS	1215.4	CSS	288.823
CV	56.0178	STD MEAN	0.113678
T:MEAN=0	21.8635	PROB>|T|	0.0001
SGN RANK	5662.5	PROB>|S|	0.0001
NUM ^= 0	150		
D:NORMAL	0.0923982	PROB>D	<0.01

QUANTILES(DEF=4)

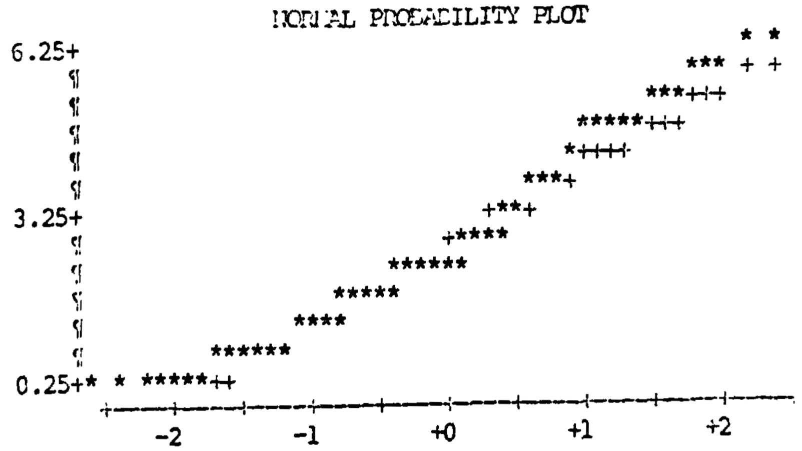
100% MAX	6.11	99%	6.0794
75% Q3	3.3425	95%	5.259
50% MED	2.18	90%	4.639
25% Q1	1.485	10%	0.734
0% MIN	0.1	5%	0.4675
		1%	0.1051
RANGE	6.01		
Q3-Q1	1.8575		
MODE	2.07		

EXTREMES

LOWEST	HIGHEST
0.1	5.57
0.11	5.61
0.34	5.88
0.37	6.05
0.4	6.11

```
STEM LEAF                              #  BOXPLOT
   6 01                                2     |
   5 669                               3     |
   5 2234                              4     |
   4 55666667789                      11     |
   4 114                               3     |
   3 555566677889                     12     |
   3 00000001222344                   14  +-----+
   2 55555566667788899                17  |     |
   2 000111111111111222233333444      27  *--+--*
   1 555556667778888899999            21  |     |
   1 0001122233334444                 16  +-----+
   0 5556667788999                    13     |
   0 1134444                           7     |
     ----+----+----+----+----+---
```

NORMAL PROBABILITY PLOT



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VARIABLE=LV

MOMENTS			
N	150	SUM WGTS	150
MEAN	879.899	SUM	131985
STD DEV	407.078	VARIANCE	165712
SKEWNESS	0.37671	KURTOSIS	-0.428885
USS	140824399	CSS	24691150
CV	46.2642	STD MEAN	33.2378
T:MEAN=0	26.4729	PROB>¦T¦	0.0001
SGN RANK	5662.5	PROB>¦S¦	0.0001
NUM ^= 0	150		
D:NORMAL	0.084998	PROB>D	<0.01

QUANTILES(DEF=4)			
100% MAX	2029.6	99%	1918.67
75% Q3	1177.37	95%	1596.11
50% MED	802.1	90%	1453.14
25% Q1	544.4	10%	389.15
0% MIN	83.6	5%	250.125
		1%	111.089
RANGE	1946		
Q3-Q1	632.975		
MODE	83.6		

EXTREMES

LOWEST	HIGHEST
83.6	1655.4
137.5	1785.3
155.2	1809.5
213.4	1812.1
233.8	2029.6

```
STEM LEAF                        #  BOXPL                 NORMAL PROBABILITY PLOT
  20 3                           1      |  2050+
  19                                    |      |
  18 11                          2      |  1850+                                         *
  17 9                           1      |      |                                        *+
  16 246                         3      |  1650+                                     ***+
  15 01177                       5      |      |                                  ***+
  14 2444589                     7      |  1450+                                **++
  13 01447788                    8      |      |                              **+
  12 00113489                    8      |  1250+                            **+
  11 0000125799                 10  +-----+    |                          **
  10 00233455899                11  |     |  1050+                      +**
   9 01233566789                11  |     |      |                    +**
   8 14455888                    8  *--+--*   850+                  +**
   7 0011122234445556667777 9999 25 |     |      |               *****
   6 1266677889                 10  |     |   650+             ***
   5 00111134567                11  +-----+      |          +***
   4 01124567889                11      |   450+         ***
   3 3345589999                 10      |      |     *****
   2 13455                       5      |   250+ ****++
   1 46                          2      |      | * * +++
   0 8                           1      |    50+* + ++
     ----+----+----+----+                      +----+----+----+----+----+----+----+----+----+----+
MULTIPLY STEM.LEAF BY 10**+02                     -2        -1        +0        +1        +2
```

The next four pages contain the summary statistics for each of the univariate variables Au, Lu, Av, and Lv, respectively. The reference altitude is

ALTITUDE - 14,000

VARIABLE=AU

```
                MOMENTS                                              QUANTILES(DEF=4)

N           150        SUM WGTS       150     100% MAX     7.76       99%      6.50538
MEAN        2.45373    SUM            368.06   75% Q3      3.265      95%      4.59
STD DEV     1.22695    VARIANCE       1.5054   50% MED     2.28       90%      3.96
SKEWNESS    0.738069   KURTOSIS       1.32473  25% Q1      1.61       10%      1.061
USS         1127.42    CSS            224.304   0% MIN     0           5%      0.6265
CV          50.0032    STD MEAN       0.10018                          1%      0.153
T:MEAN=0    24.4933    PROB>|T|       0.0001   RANGE       7.76
SGN RANK    5587.5     PROB>|S|       0.0001   Q3-Q1       1.655
NUM ^= 0    149                                MODE        2.28
D:NORMAL    0.0902606  PROB>D         <0.01

              EXTREMES

      LOWEST            HIGHEST
           0               4.79
         0.3               5.03
        0.41                5.2
        0.45                5.3
        0.45               7.76

STEM LEAF                                     #  BOXPLOT
   7 8                                        1     0
   7
   6
   6
   5
   5 023                                      3     |
   4 55778                                    5     |
   4 0001223                                  7     |
   3 55577777788899999                       17     |
   3 0000111222223334                        16  +-----+
   2 555556777999999                         15  |     |
   2 0000000111223333333333333444            28  *--+--*
   1 55555556666677777788888888999           29  +-----+
   1 0111111222333444                        16     |
   0 56677889                                 8     |
   0 03444                                    5     |
     ----+----+----+----+----+----

                                                        NORMAL PROBABILITY PLOT
                                              7.75+                                                  *
                                                  |
                                                  |
                                              6.25+
                                                  |                                                  +
                                                  |                                              ** *
                                              4.75+                                          ****+
                                                  |                                      +**+
                                                  |                                 ******
                                              3.25+                             ***+
                                                  |                         +****
                                                  |                     +*****
                                              1.75+                ******
                                                  |           *****+
                                                  |       ****++
                                              0.25+* * ****++
                                                   +----+----+----+----+----+----+----+----+----+----+
                                                       -2        -1        +0        +1        +2
```

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VARIABLE=LU

MOMENTS

N	150	SUM WGTS	150
MEAN	903.229	SUM	135484
STD DEV	417.218	VARIANCE	174071
SKEWNESS	0.812246	KURTOSIS	0.412736
USS	148309808	CSS	25936505
CV	46.1918	STD MEAN	34.0657
T:MEAN=0	26.5143	PROB>/T/	0.0001
SGN RANK	5662.5	PROB>/S/	0.0001
NUM ^= 0	150		
D:NORMAL	0.0964196	PROB>D	<0.01

QUANTILES(DEF=4)

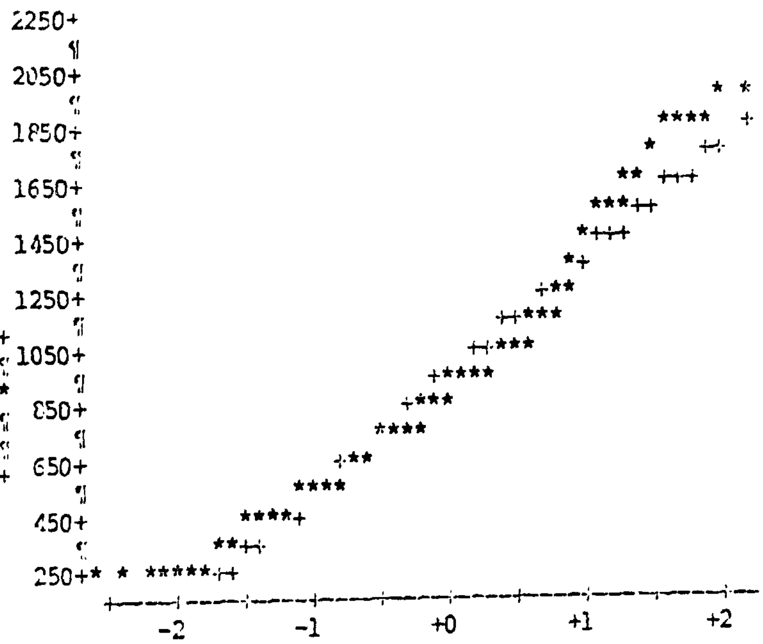
100% MAX	2294	99%	2122.28
75% Q3	1[illegible]7.6	95%	1799.22
50% MED	857.45	90%	1528.01
25% Q1	589.85	10%	422.14
0% MIN	250.8	5%	303.975
		1%	251.157
RANGE	2043.2		
Q3-Q1	487.75		
MODE	250.8		

EXTREMES

LOWEST	HIGHEST
250.8	1839.3
251.5	1851.8
267.1	1929
267.2	1957.3
290.1	2294

```
STEM LEAF                      #  BOXPLOT
  22 9                         1     0
  21
  20
  19 36                        2     0
  18 1245                      4     0
  17 0019                      4     |
  16 7                         1     |
  15 02345                     5     |
  14 688                       3     |
  13 24689                     5     |
  12 5668                      4     |
  11 0124499                   7     |
  10 1223355677888            13  +-----+
   9 00122223345556667789     20  |  +  |
   8 1122355668889            13  *-----*
   7 000112233344446678899    21  |     |
   6 055555788                 9  |     |
   5 2223456667799            13  +-----+
   4 1222233336789            13     |
   3 0011689                   7     |
   2 55779                     5     |
     ----+----+----+----+
  MULTIPLY STEM.LEAF BY 10**+02
```

NORMAL PROBABILITY PLOT



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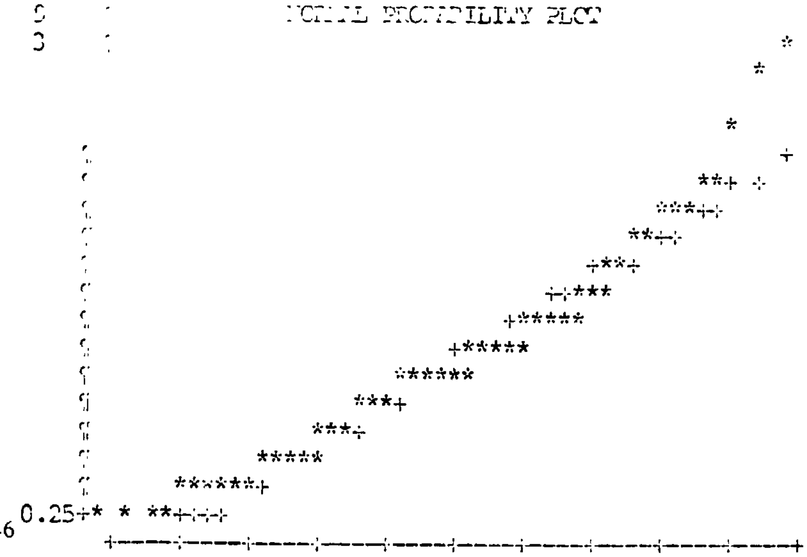
```
VARIABLE=AV

                  MOMENTS                                          QUANTILES(DEF=4)

N               150   SUM WGTS        150   100% MAX      8.75      99%     8.42869
MEAN         2.8624   SUM          429.36    75% Q3       3.58      95%       5.929
STD DEV      1.5038   VARIANCE    2.26141    50% MED      2.77      90%        4.84
SKEWNESS   0.952836   KURTOSIS     1.8421    25% Q1      1.885      10%       1.041
USS         1565.95   CSS         336.951     0% MIN         0       5%      0.7665
CV          52.5363   STD MEAN   0.122785                             1%   0.0407999
T:MEAN=0    23.3123   PROB>|T|     0.0001    RANGE        8.75
SGN RANK     5587.5   PROB>|S|     0.0001    Q3-Q1       1.695
NUM ^= 0        149                          MODE         1.95
D:NORMAL  0.0852261   PROB>D        >0.01

                EXTREMES

    LOWEST             HIGHEST
         0                6.05
      0.08                6.26
      0.25                7.21
      0.48                8.12
      0.63                8.75

STEM LEAF                                           #  BOXPLOT
   8 7                                              1     *
   8 1                                              1     0
   7
   7 2                                              1     0
   6
   6 03                                             2     0
   5 6999                                           4     |
   5 0112                                           4     |
   4 66789                                          5     |
   4 01112334                                       8     |
   3 55555666677788999                             17  +-----+
   3 000000111122344444444                         21  |     |
   2 5555555666667777788888899999                  28  *--+--*
   2 00000112222333334                             17  |     |
   1 5555666778889999                              16  +-----+
   1 0000111122344                                 13     |
   0 567788999                                      9     |
   0 012                                            3     |
     ----+----+----+----+----+---
```

NORMAL PROBABILITY PLOT



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VARIABLE=LV

MOMENTS			
N	150	SUM WGTS	150
MEAN	858.725	SUM	128809
STD DEV	373.779	VARIANCE	139711
SKEWNESS	0.656117	KURTOSIS	0.273074
USS	131428326	CSS	20816946
CV	43.5272	STD MEAN	30.519
T:MEAN=0	28.1374	PROB>|T|	0.0001
SGN RANK	5662.5	PROB>|S|	0.0001
NUM ^= 0	150		
D:NORMAL	0.0617395	PROB>D	^0.15

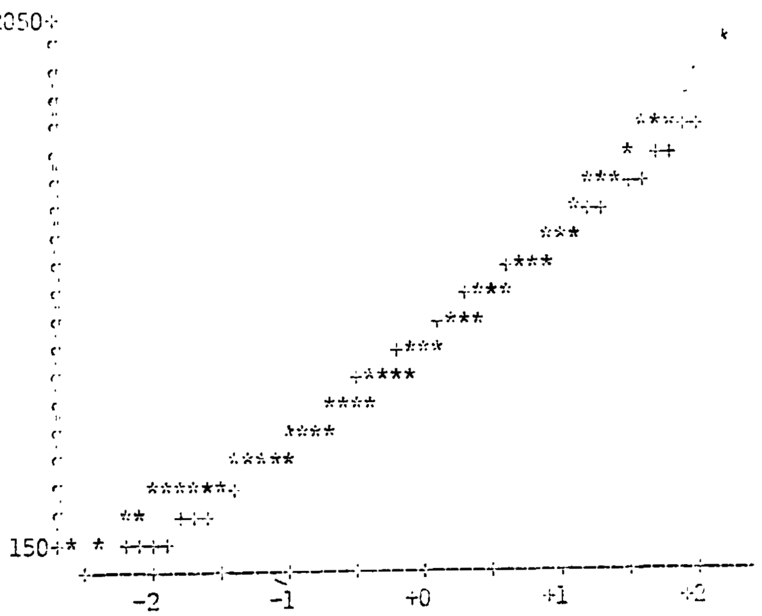
QUANTILES(DEF=4)			
100% MAX	1996.4	99%	1988.75
75% Q3	1083.62	95%	1566.57
50% MED	826.65	90%	1385.02
25% Q1	584.825	10%	421.8
0% MIN	143.7	5%	343.3
		1%	161.346
RANGE	1852.7		
Q3-Q1	498.8		
MODE	842		

EXTREMES

LOWEST	HIGHEST
143.7	1644.2
178.3	1741.5
222.5	1836
277.8	1981.4
323.3	1996.4

```
STEM LEAF                     #  BOXPLOT
  20 0                        1
  19 8                        1     0
  18 4                        1     0
  17 4                        1     |
  16 124                      3     |
  15 23                       2     |
  14 34678                    5     |
  13 7789                     4     |
  12 0234679                  7     |
  11 014457899               10     |
  10 1112336678899           13  +-----+
   9 00113346666779          14  |     |
   8 00024445556667889       17  *--+--*
   7 011223346678888         15  |     |
   6 004456678999999         15  |     |
   5 00444555699999          14  +-----+
   4 002233346778899         15     |
   3 24445579                 8     |
   2 28                       2     |
   1 48                       2     |
     ----+----+----+----+
MULTIPLY STEM.LEAF BY 10**+02
```

NORMAL PROBABILITY PLOT



APPENDIX B

Plots for the bivariate gamma density function, scatter plots, and contours are given for each data set.

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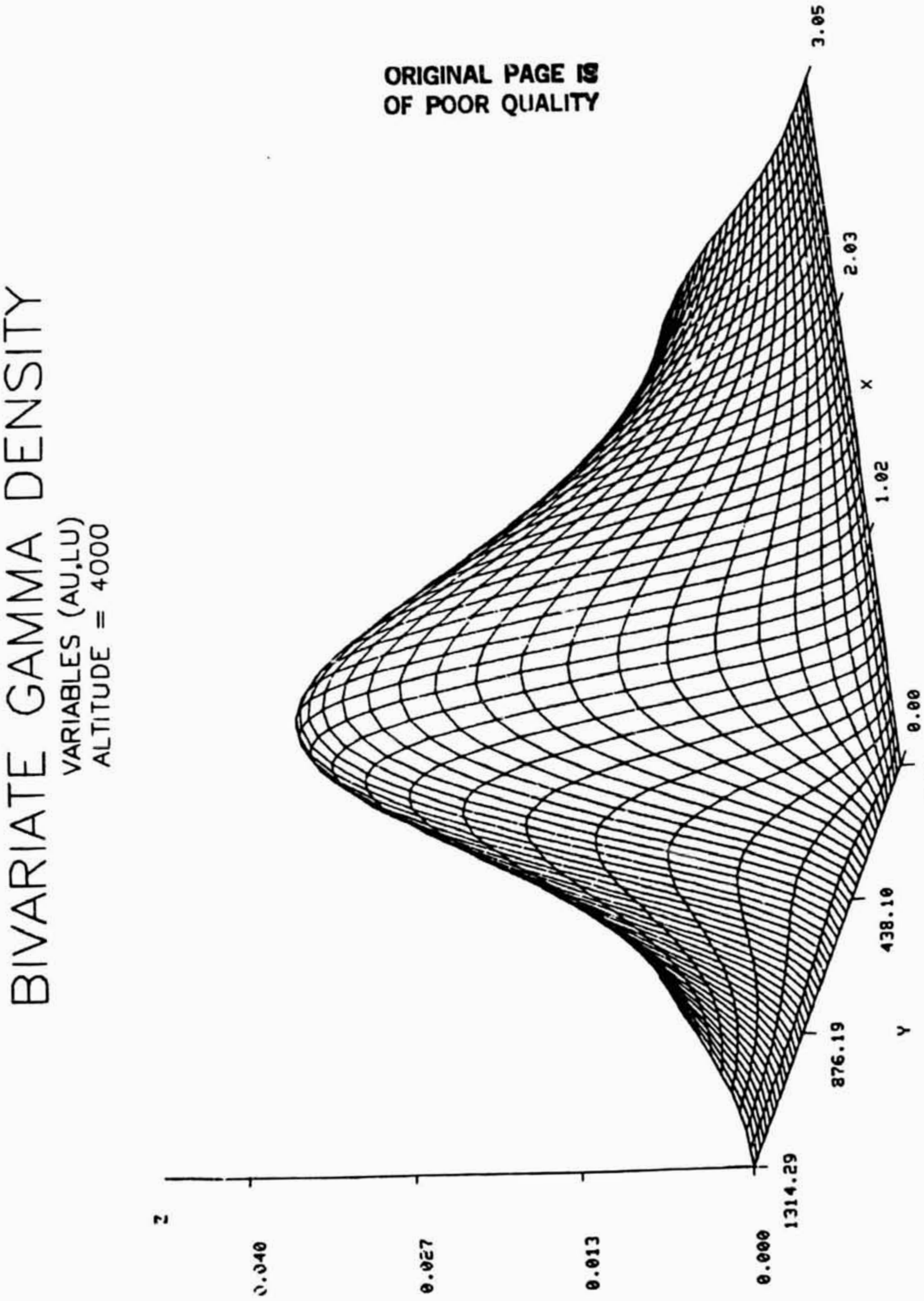

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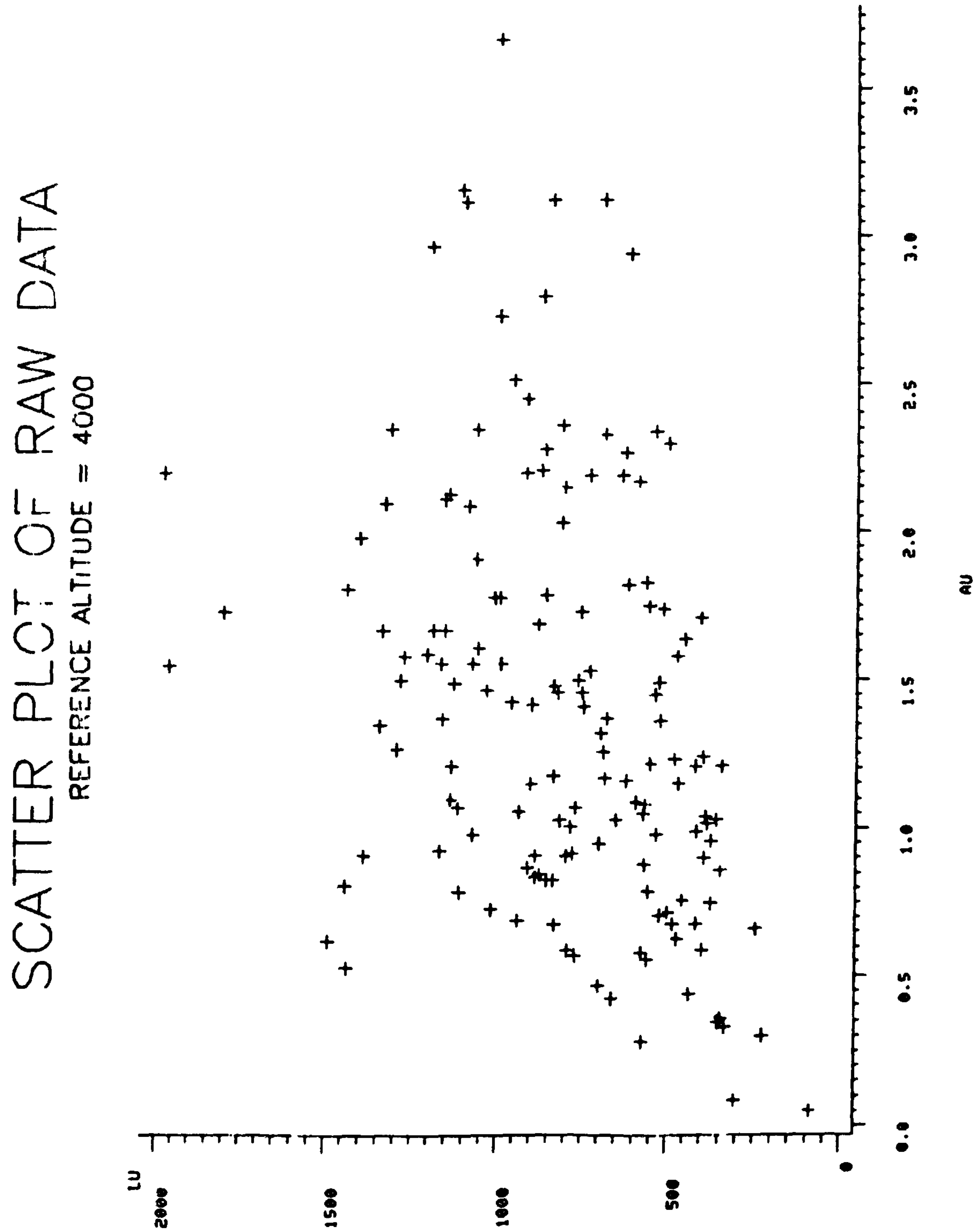

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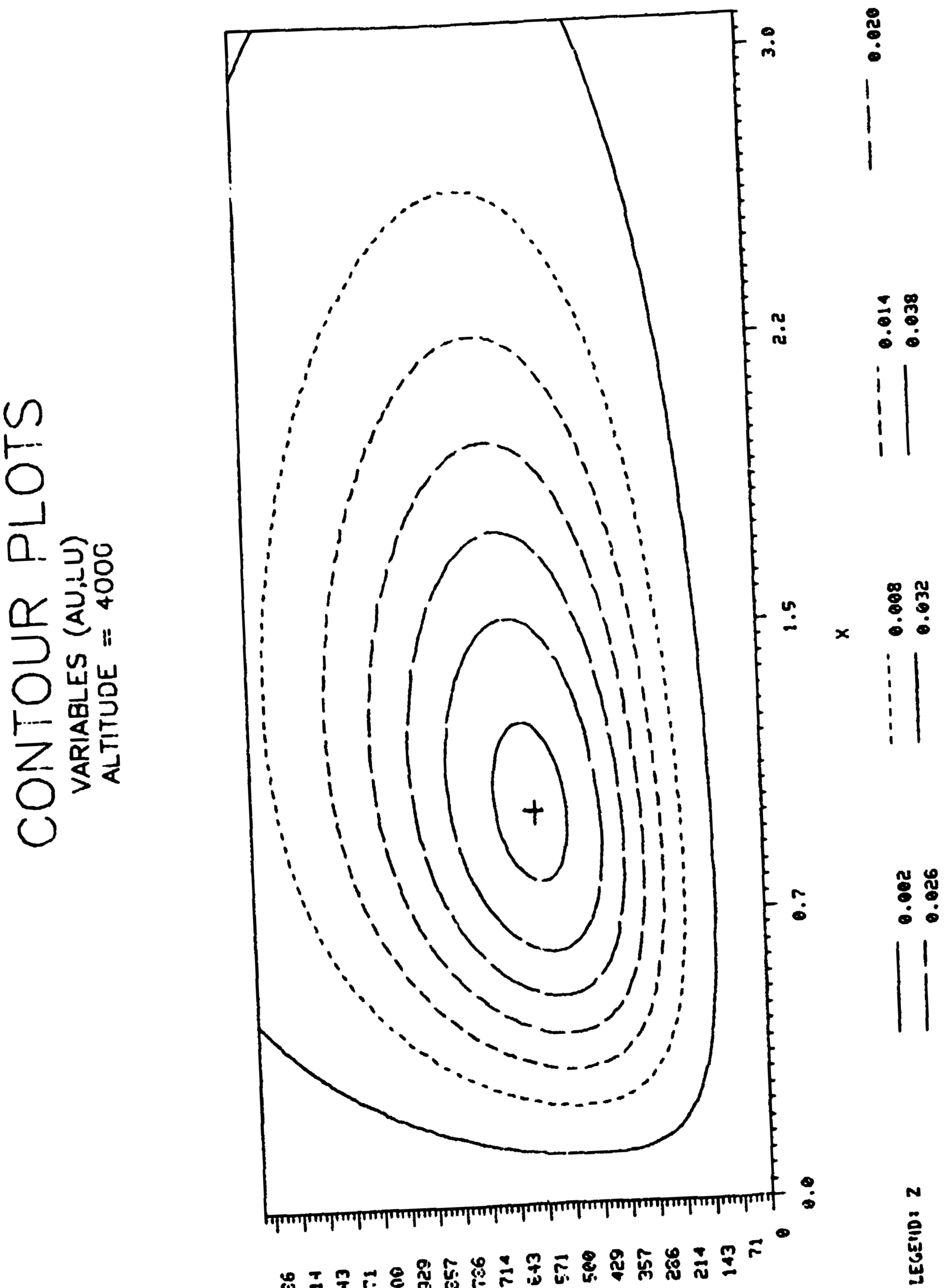

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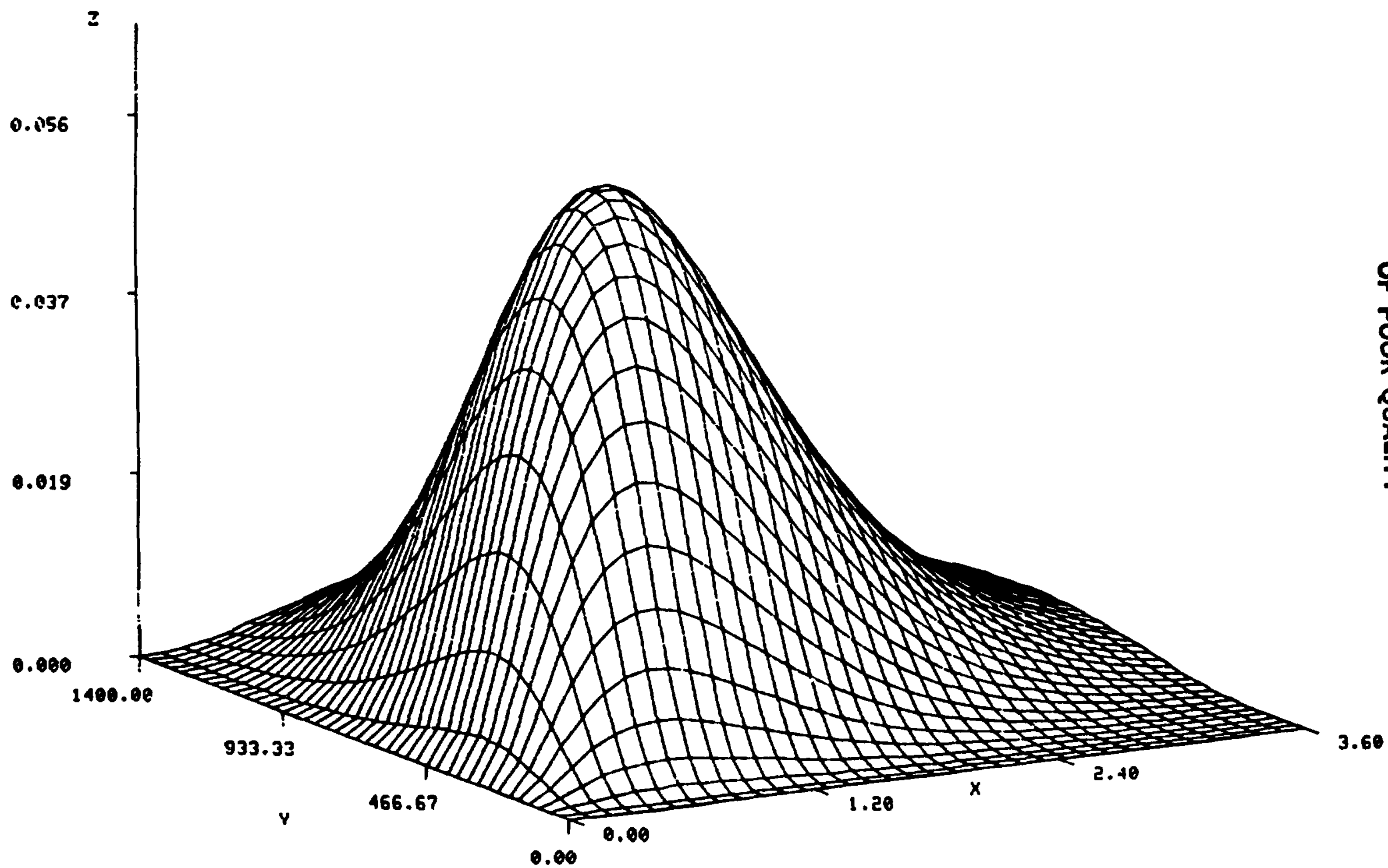
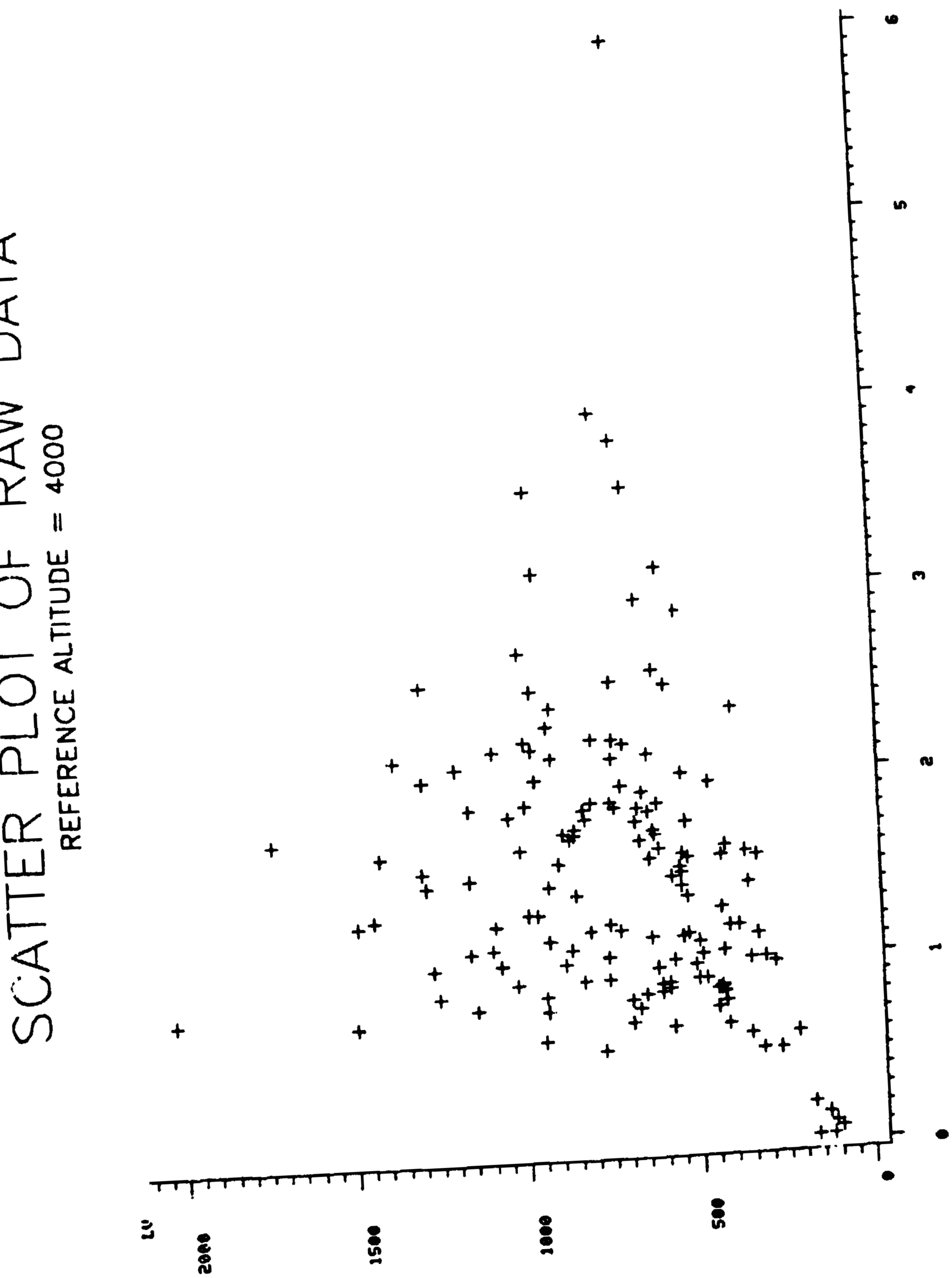
BIVARIATE GAMMA DENSITY

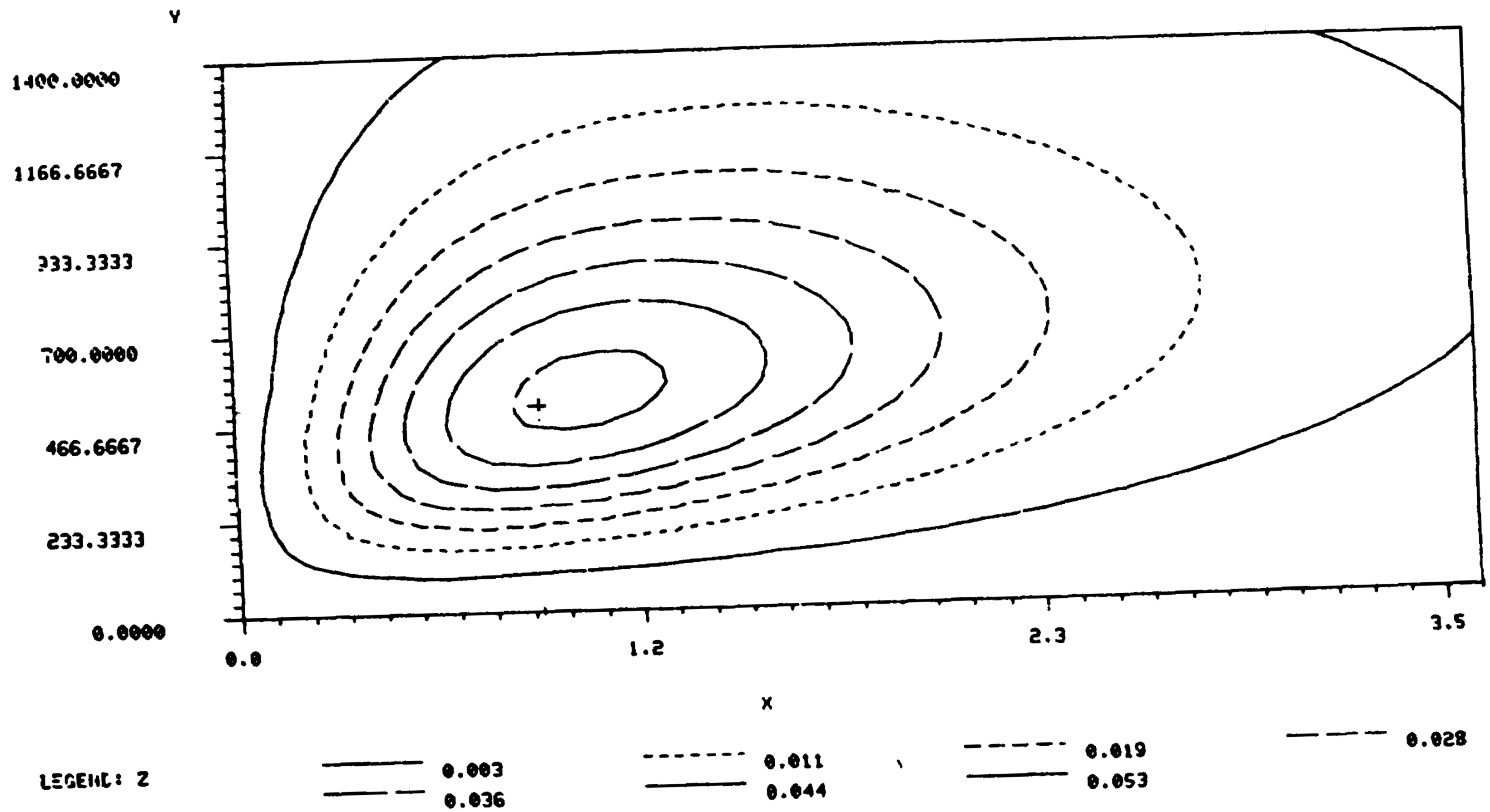
VARIABLES (AV,LV)
ALTITUDE = 4000

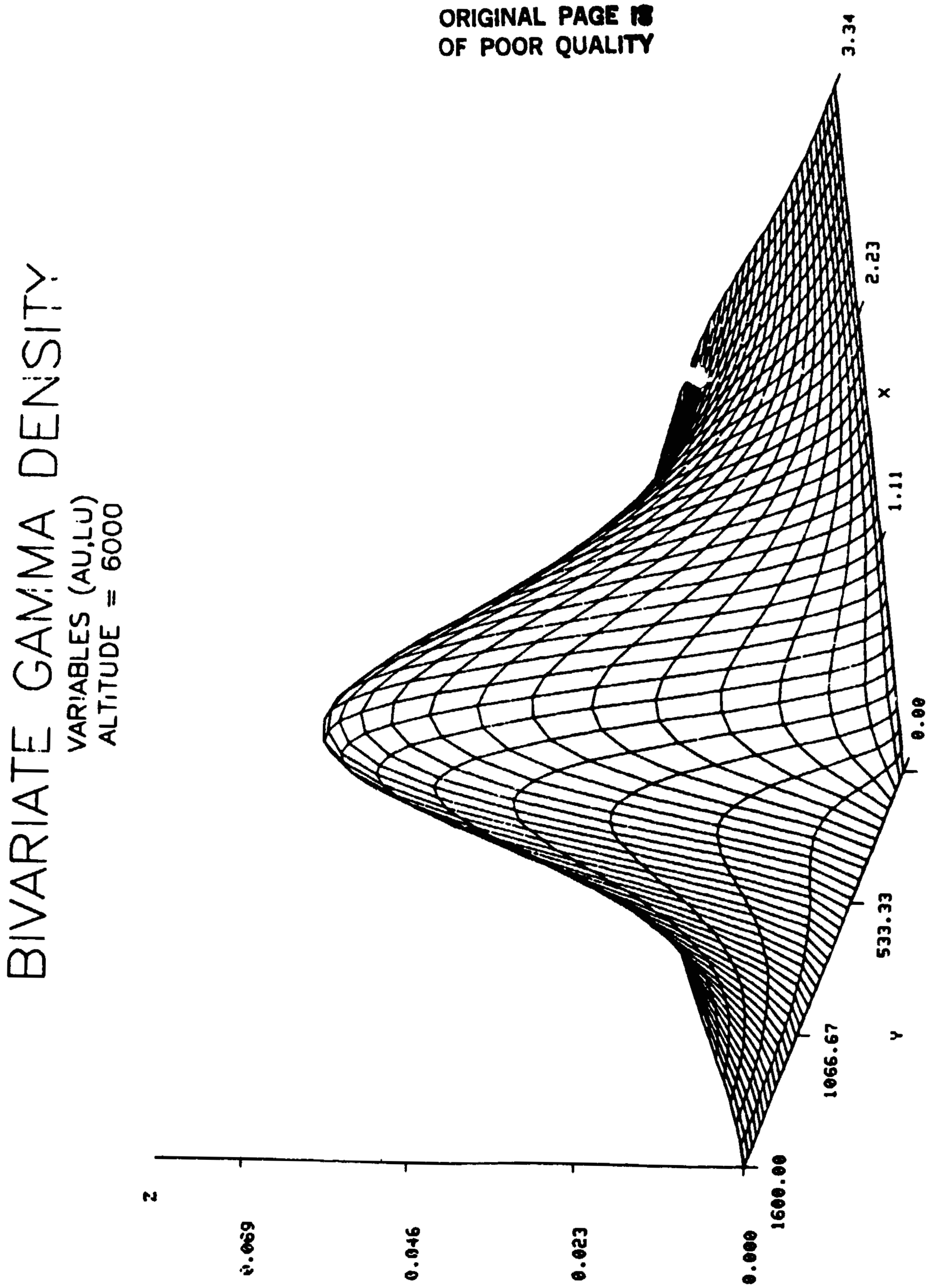



SCATTER PLOT OF RAW DATA
REFERENCE ALTITUDE = 4000
0
1
2
3
4
5
6
500
1000
1500
2000

CONTOUR PLOTS

VARIABLES (AV,LV)
ALTITUDE = 4000




BIVARIATE GAMMA DENSITY
VARIABLES (AU,LU)
ALTITUDE = 6000
Z
0.069
0.046
0.023
0.000
1600.00
1066.67
Y
533.33
0.00
0.00
1.11
X
2.23
3.34

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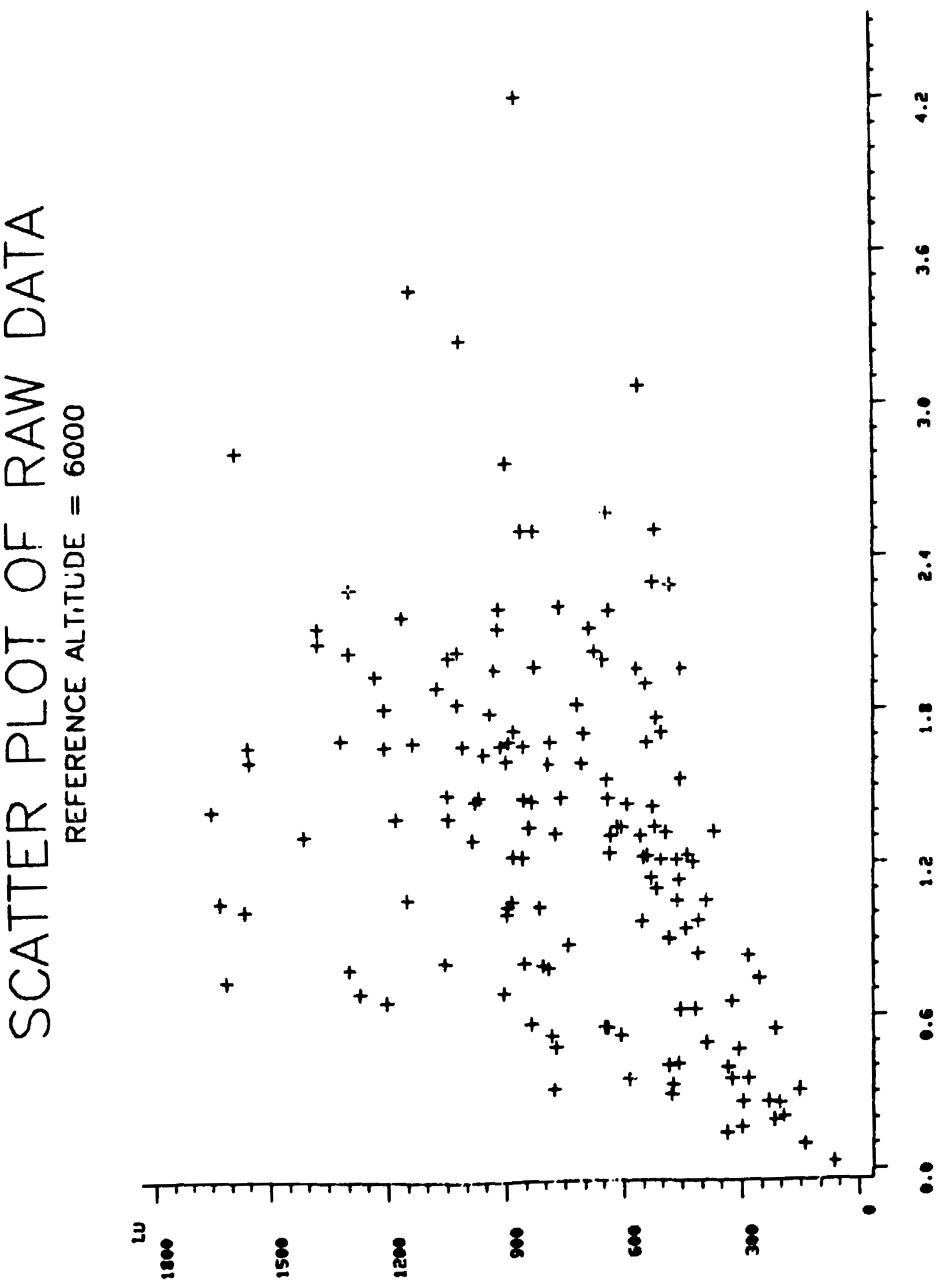
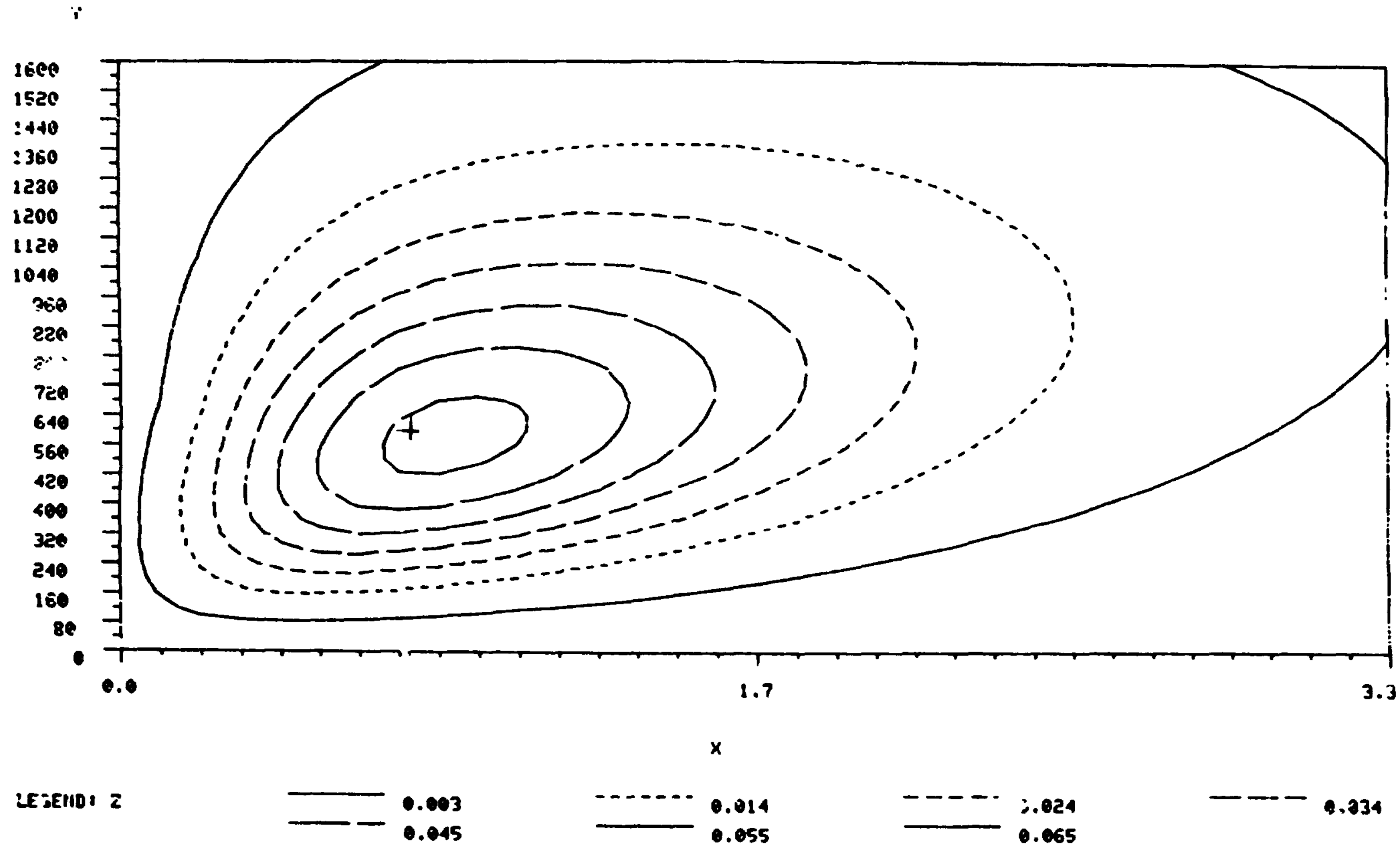
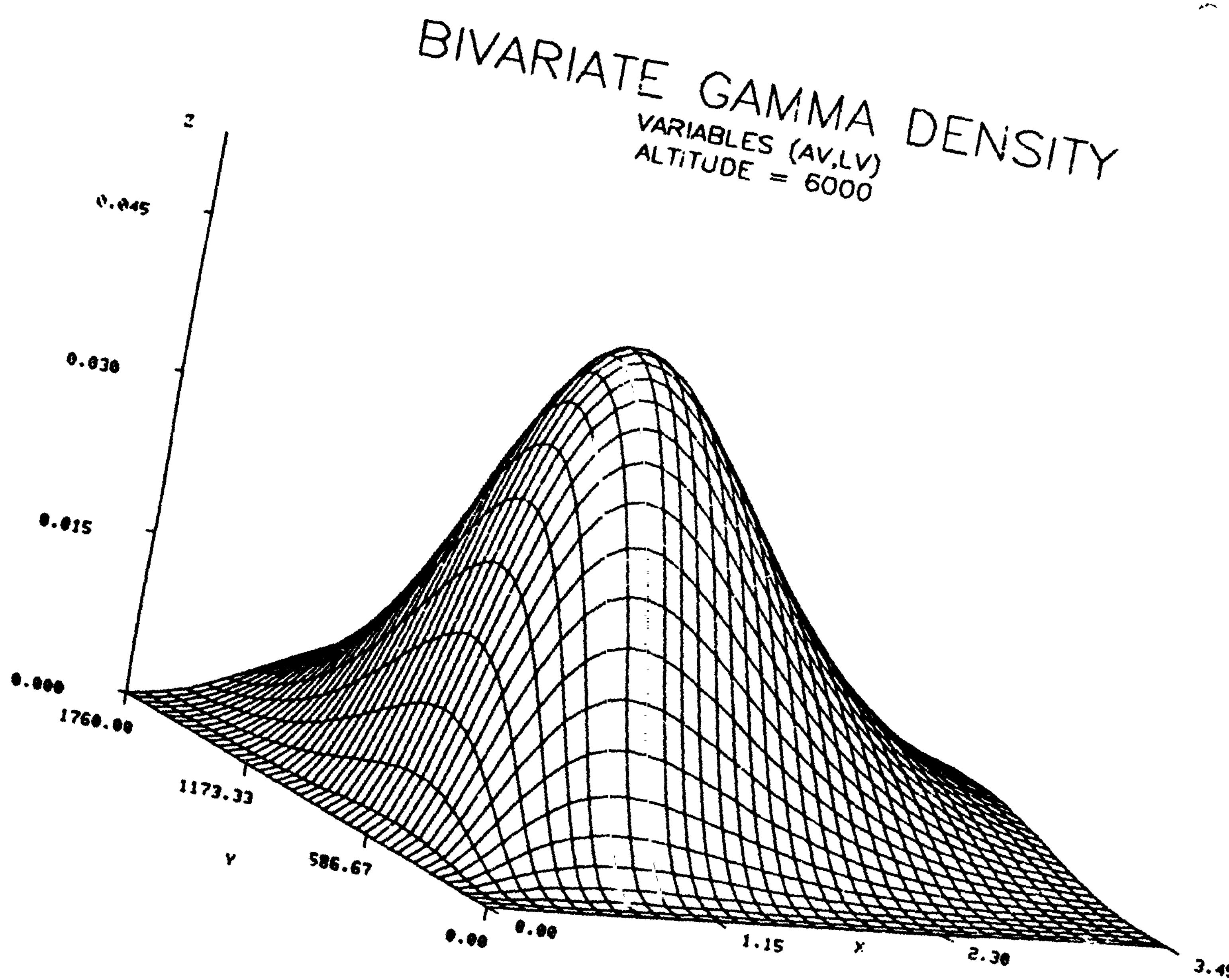


CONTOUR PLOTS

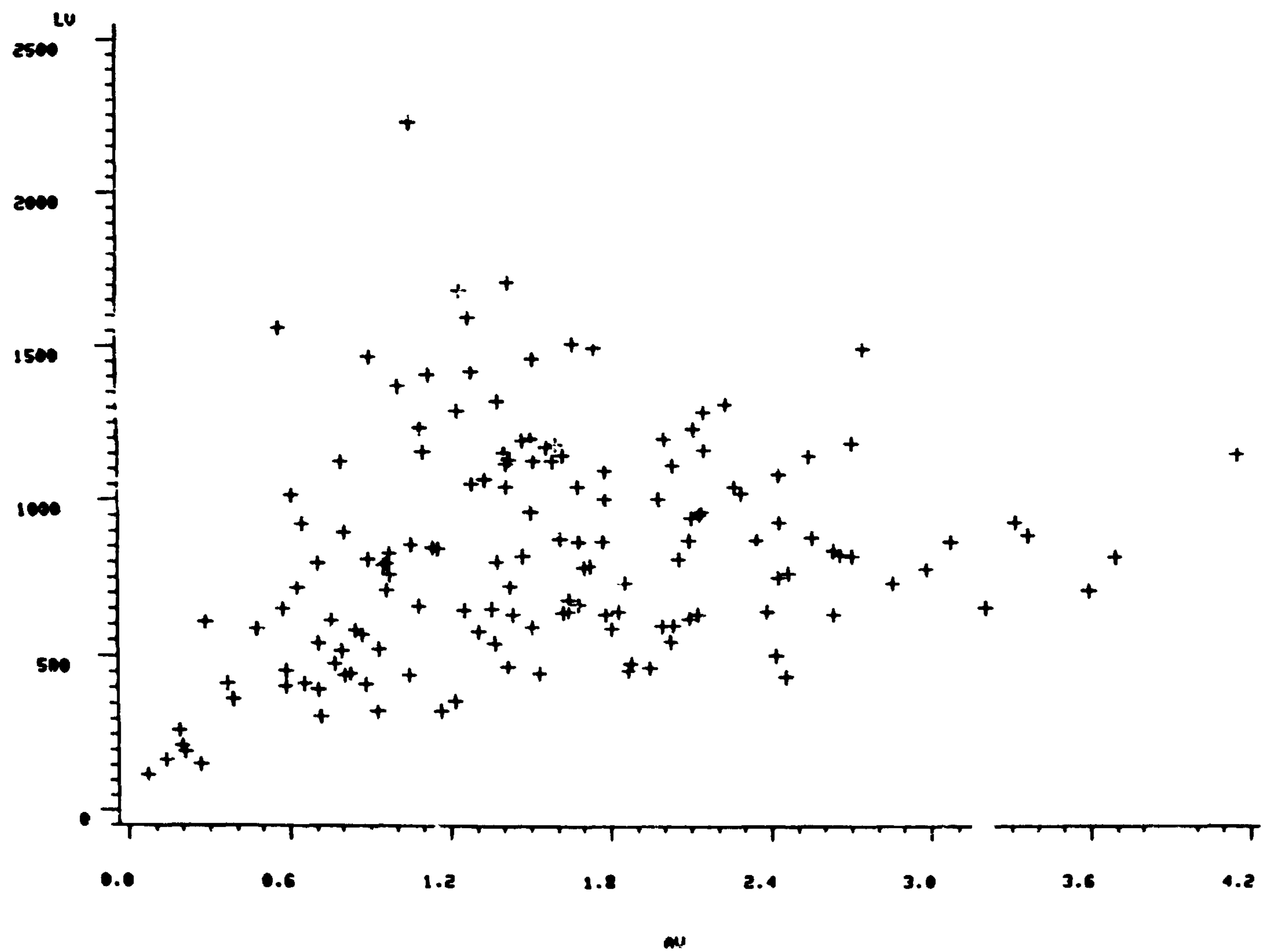
VARIABLES (AU,LU)
ALTITUDE = 6000




BIVARIATE GAMMA DENSITY
VARIABLES (AV,LV)
ALTITUDE = 6000
Z
0.045
0.030
0.015
0.000
1760.00
1173.33
Y
586.67
0.00
0.00
1.15
X
2.30
3.45

SCATTER PLOT OF RAW DATA

REFERENCE ALTITUDE = 6000



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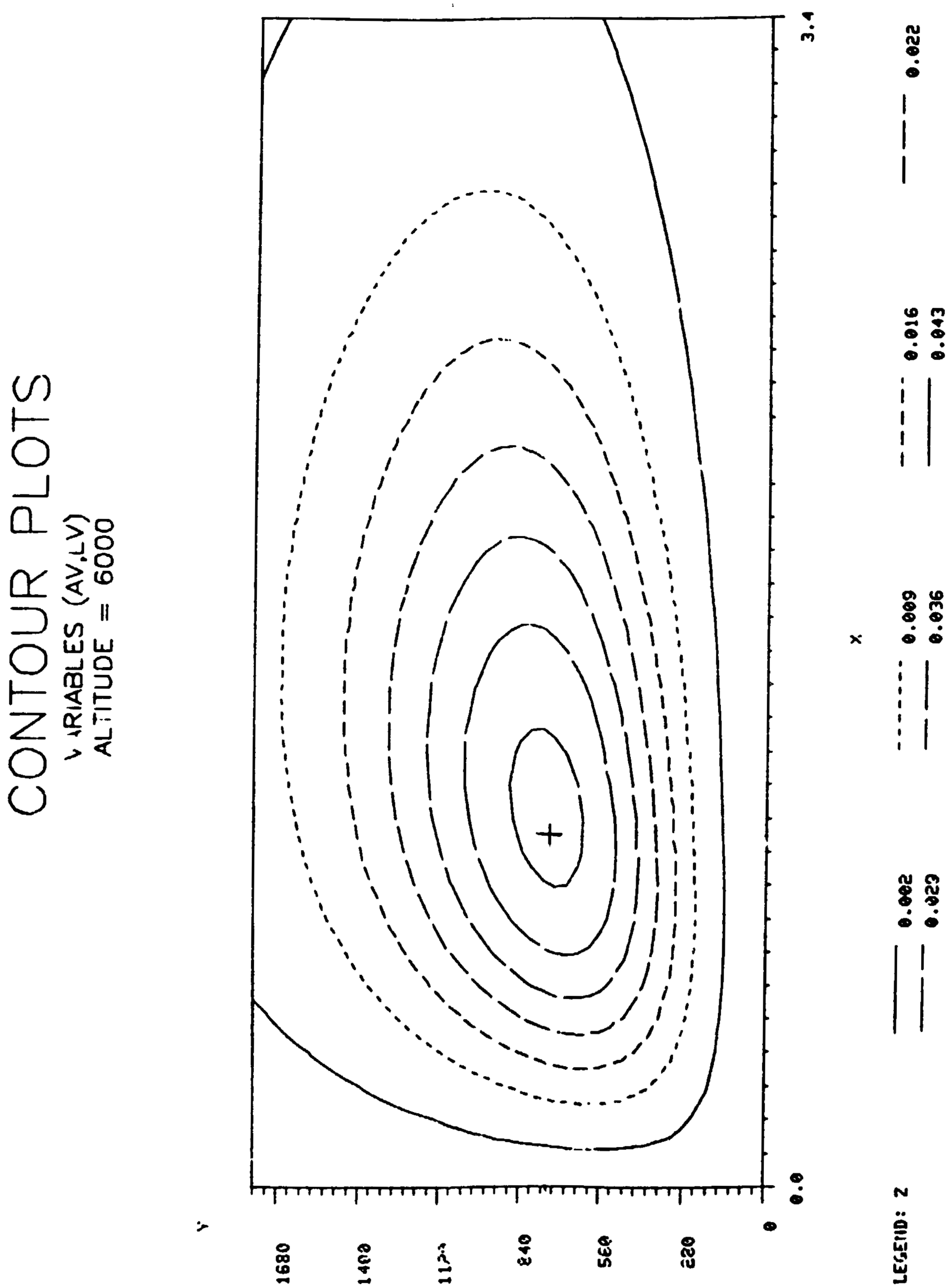
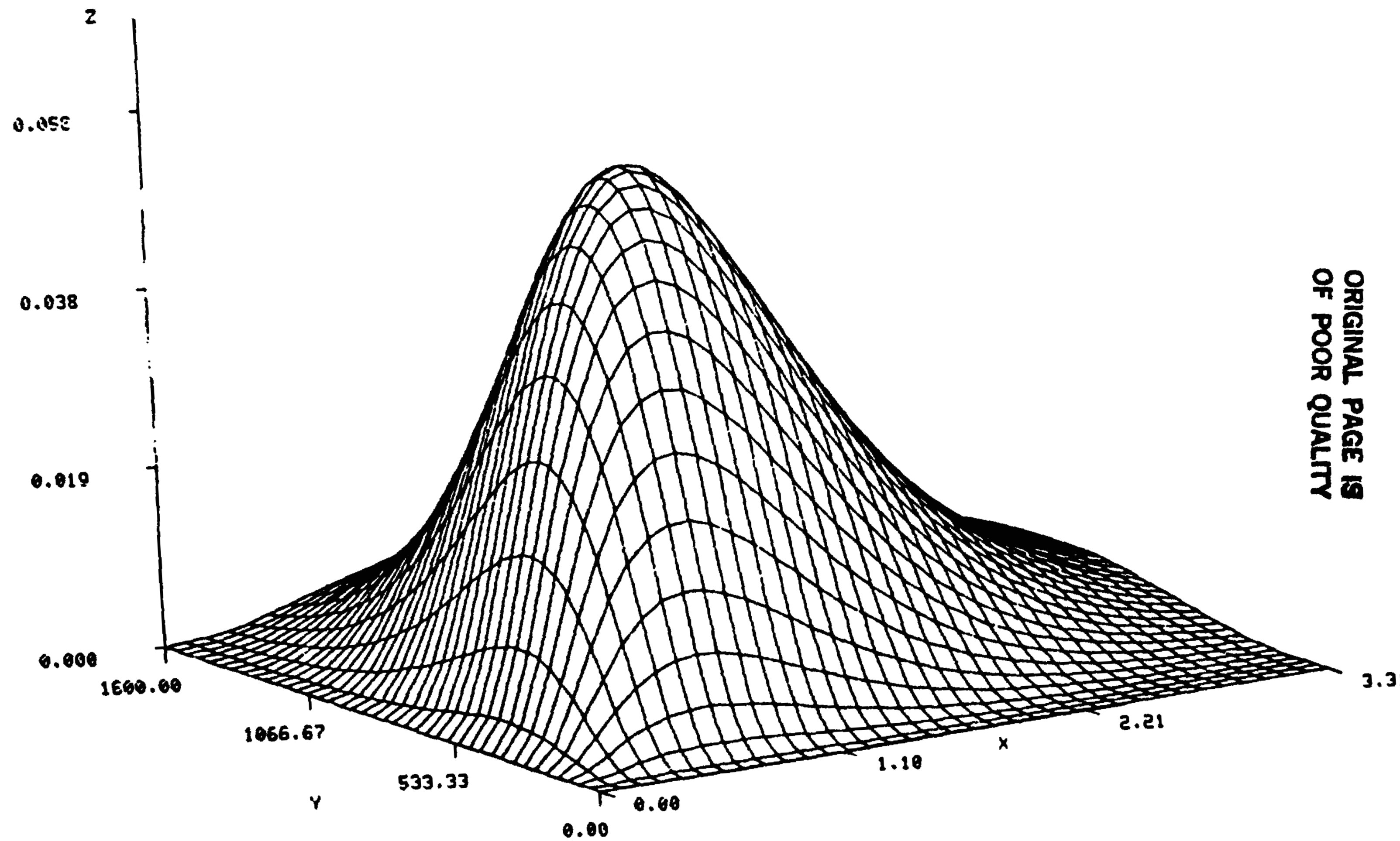


BIVARIATE GAMMA DENSITY

VARIABLES (AU,LU)
ALTITUDE = 8000



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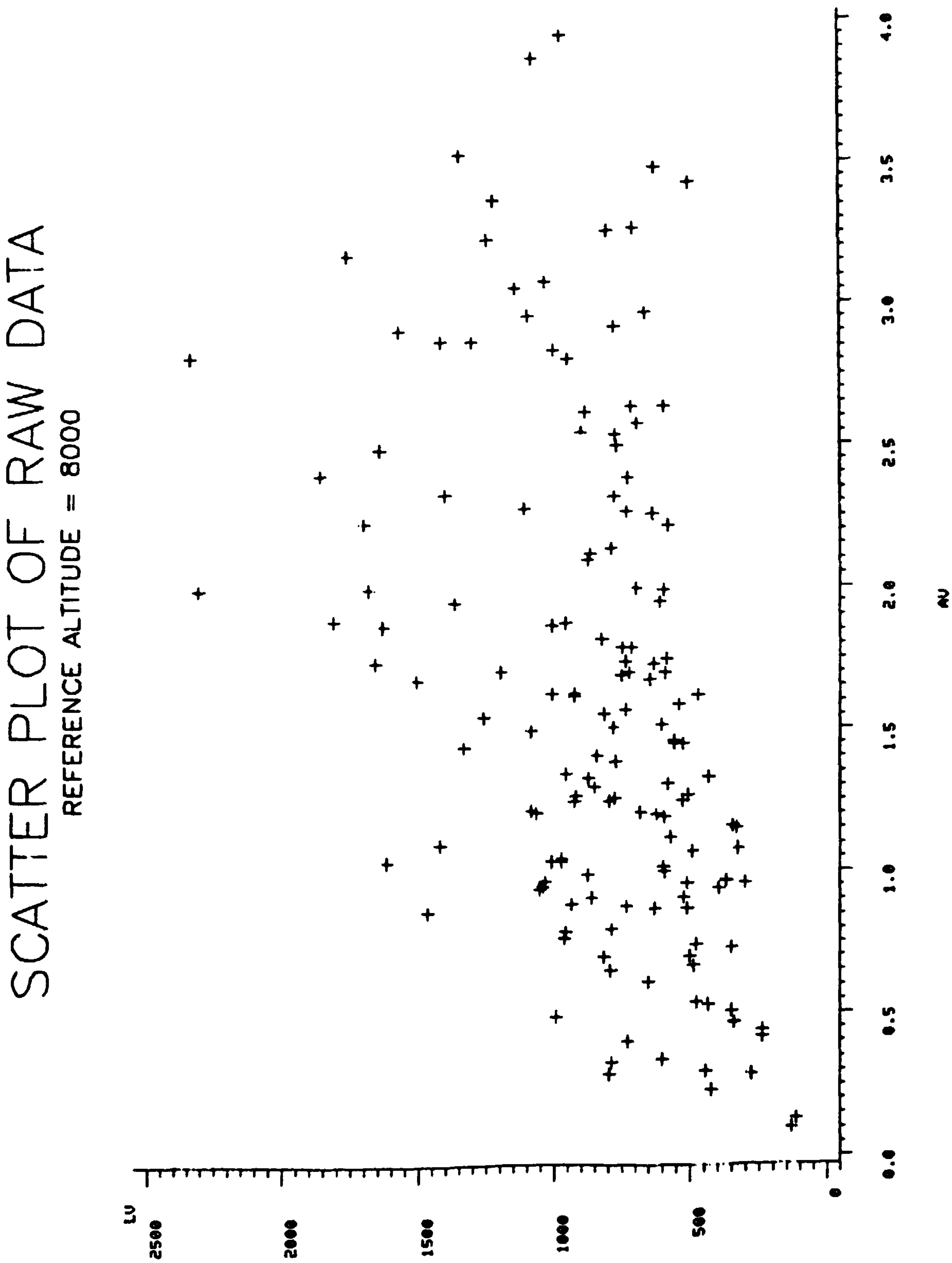

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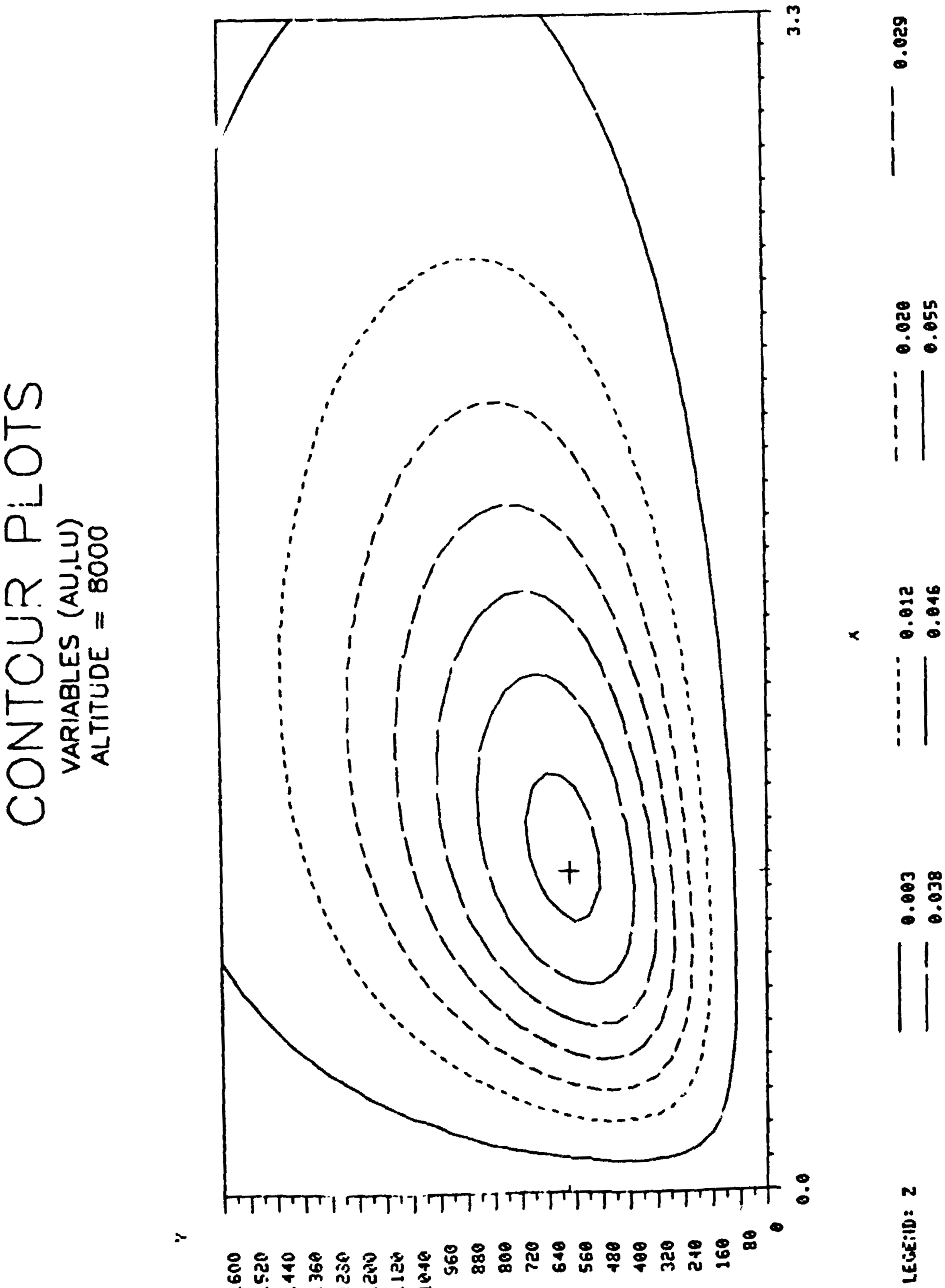
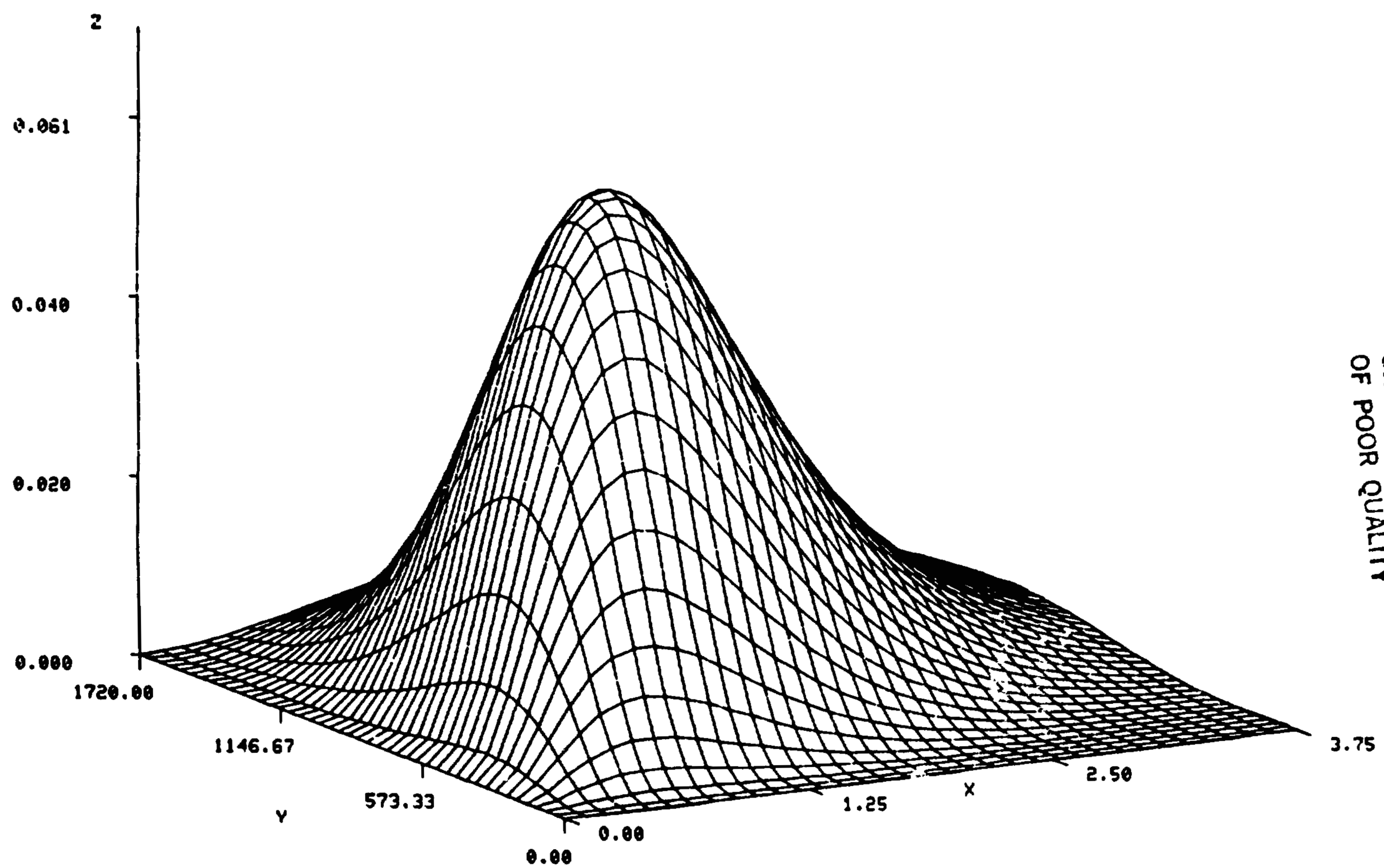


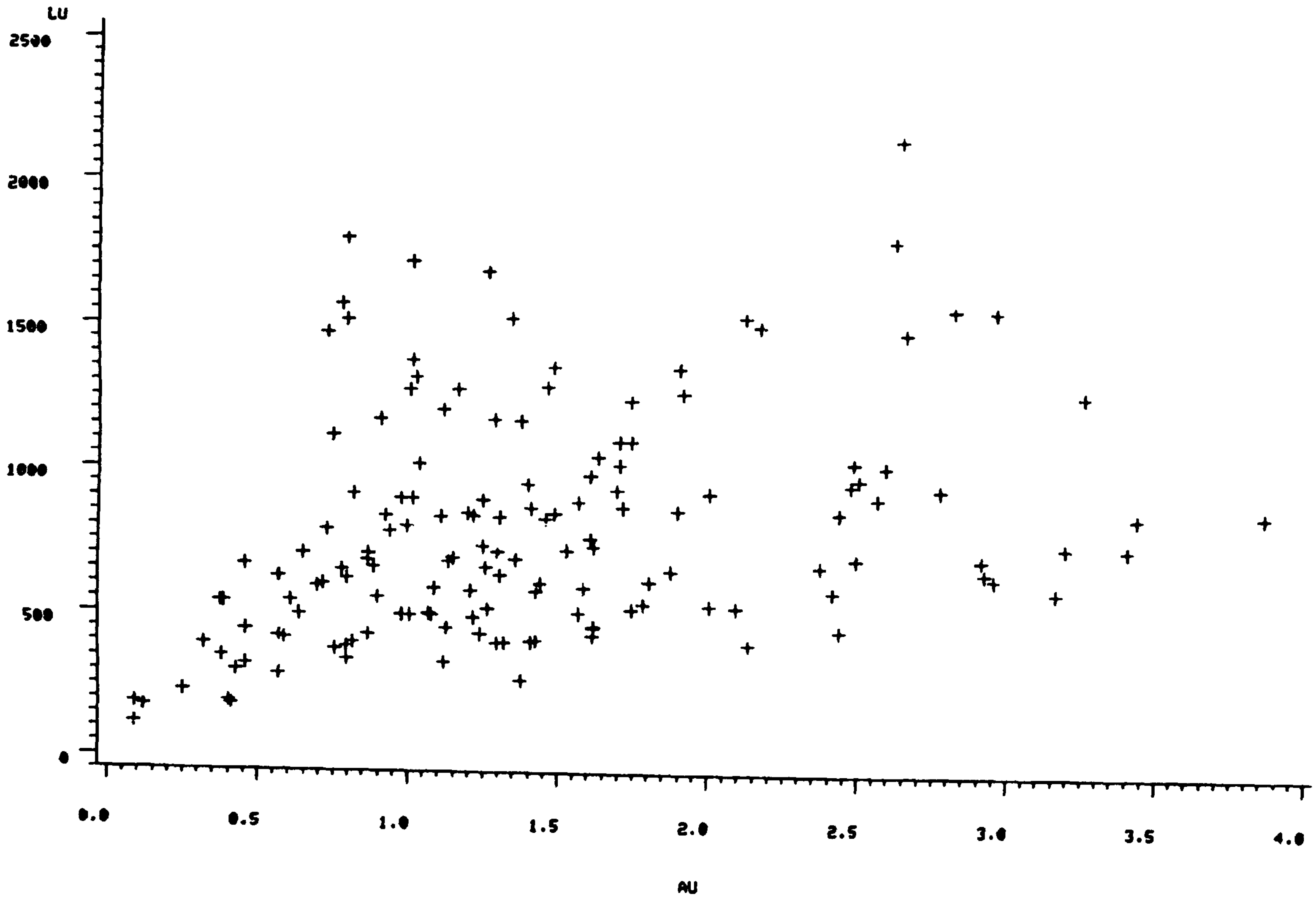
BIVARIATE GAMMA DENSITY

VARIABLES (AV,LV)
ALTITUDE = 8000



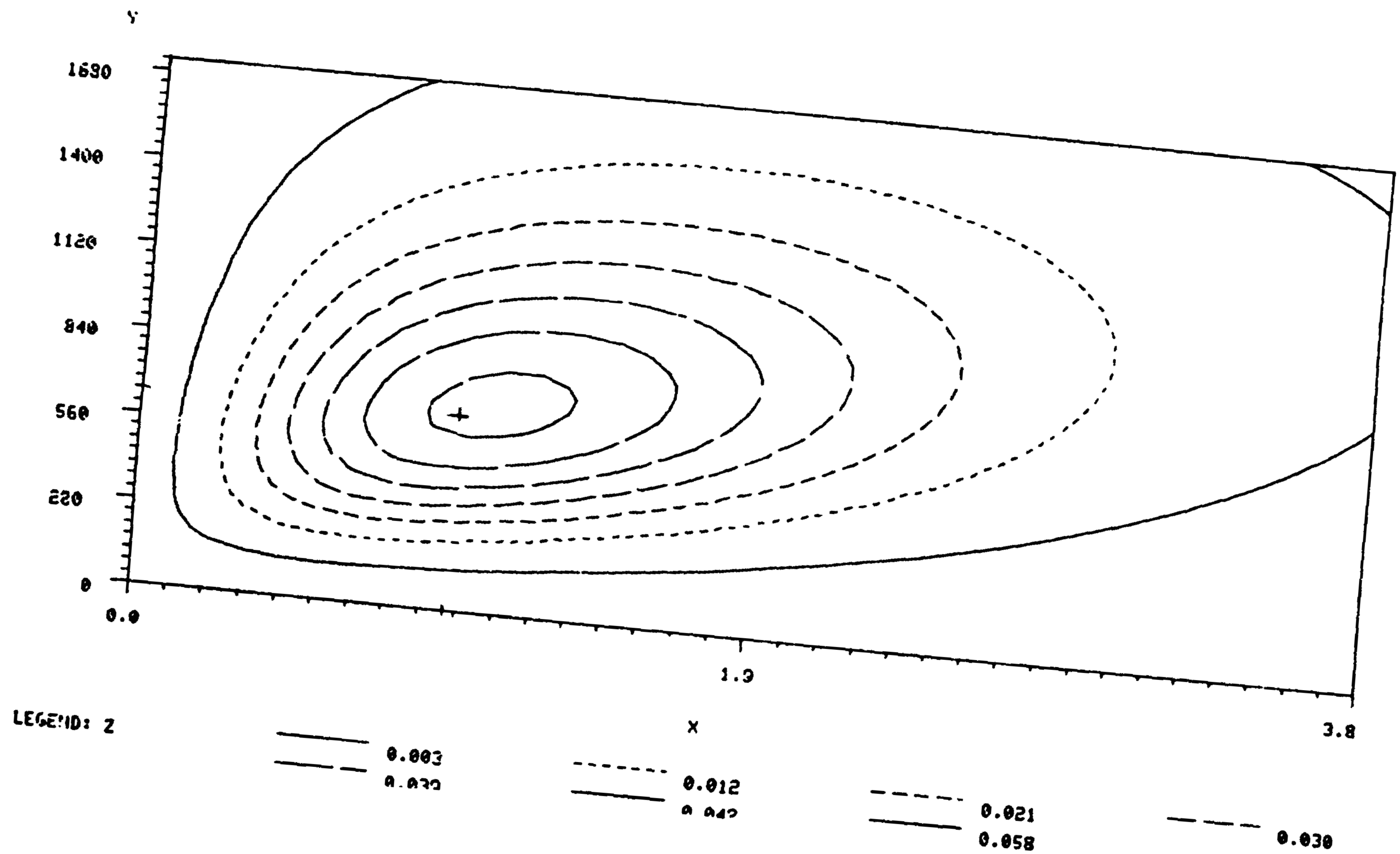
SCATTER PLOT OF RAW DATA

REFERENCE ALTITUDE = 8000



CONTOUR PLOTS

VARIABLES (AV,LV)
ALTITUDE = 8000



BIVARIATE GAMMA DENSITY

VARIABLES (AU,LU)
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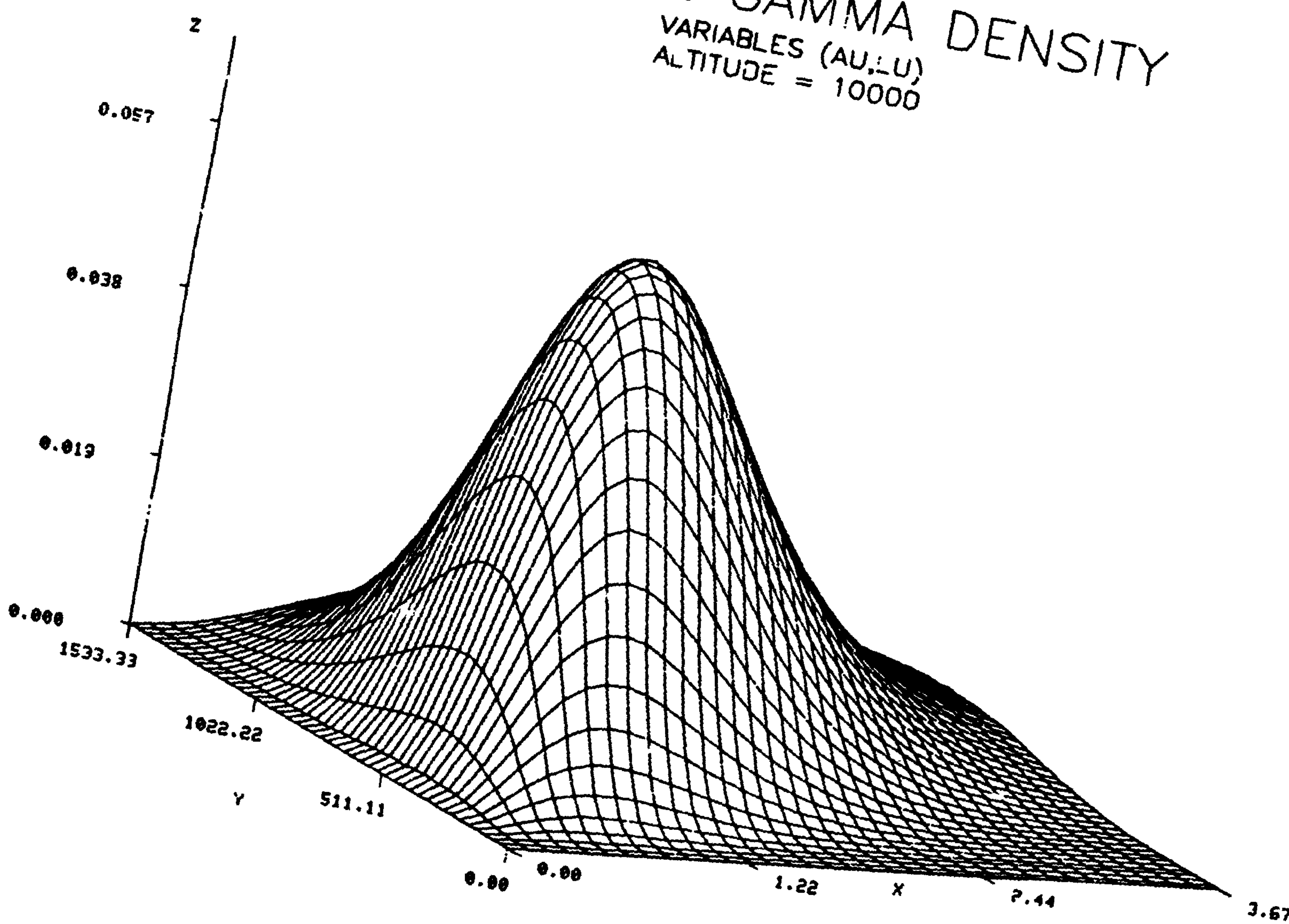

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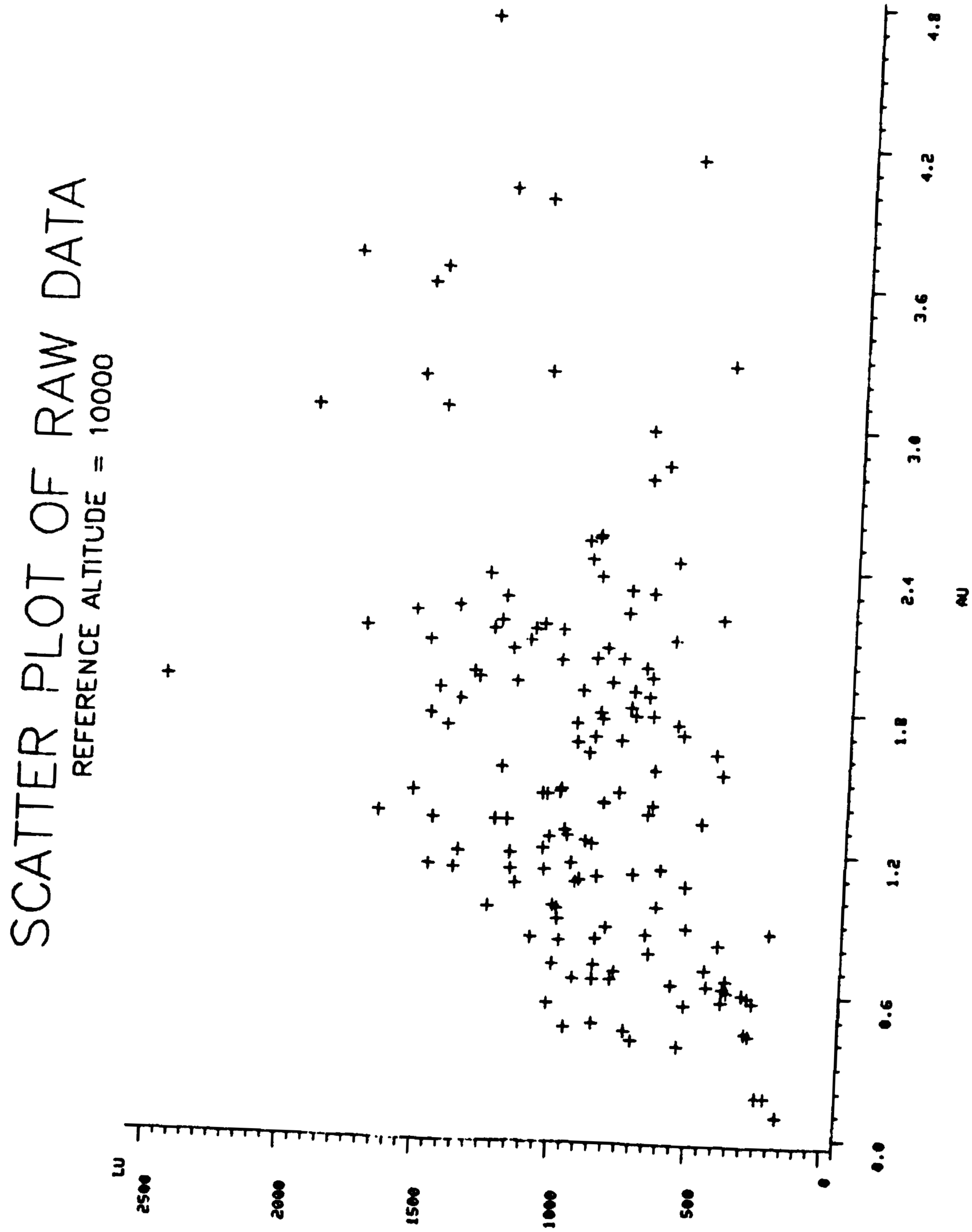
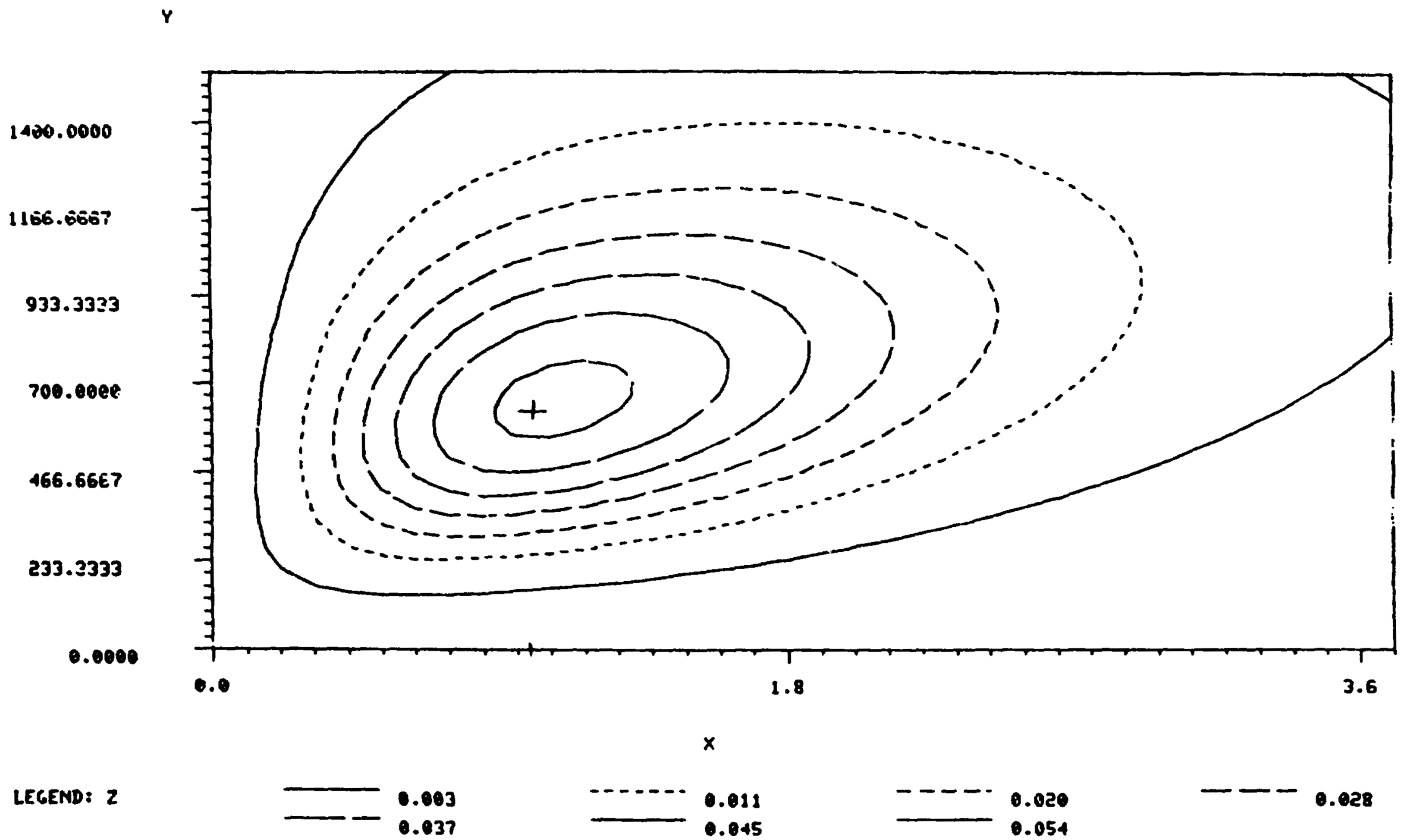
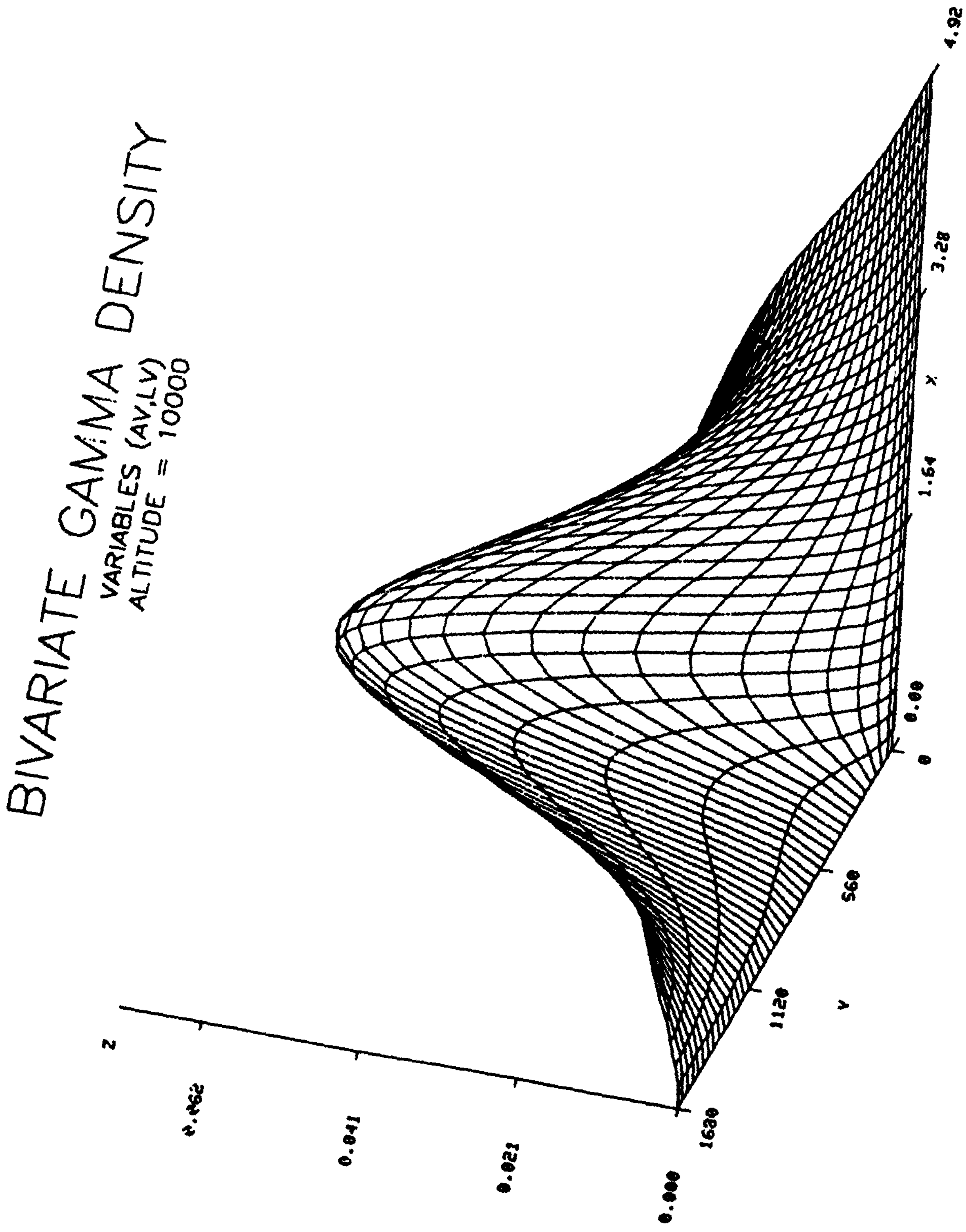


CONTOUR PLOTS

VARIABLES (AU,LU)
ALTITUDE = 10000




BIVARIATE GAMMA DENSITY
VARIABLES (AV,LV)
ALTITUDE = 10000
Z
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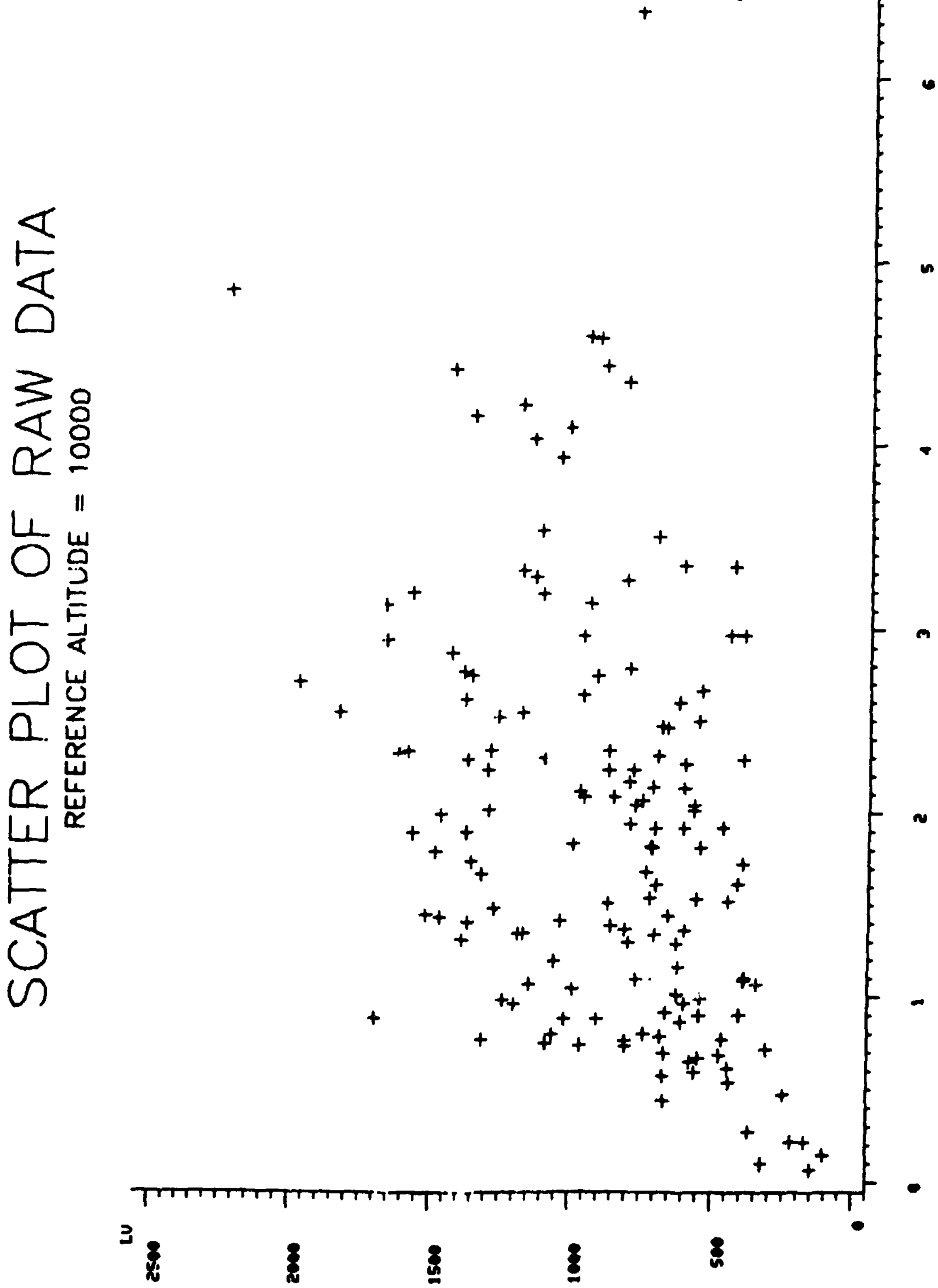
0.021
0.000
1630
1120
Y
560
0
0.00
1.64
X
3.28
4.92


SCATTER PLOT OF RAW DATA
REFERENCE ALTITUDE = 10000
LU
2500
2000
1500
1000
500
0
AV
0
1
2
3
4
5
6

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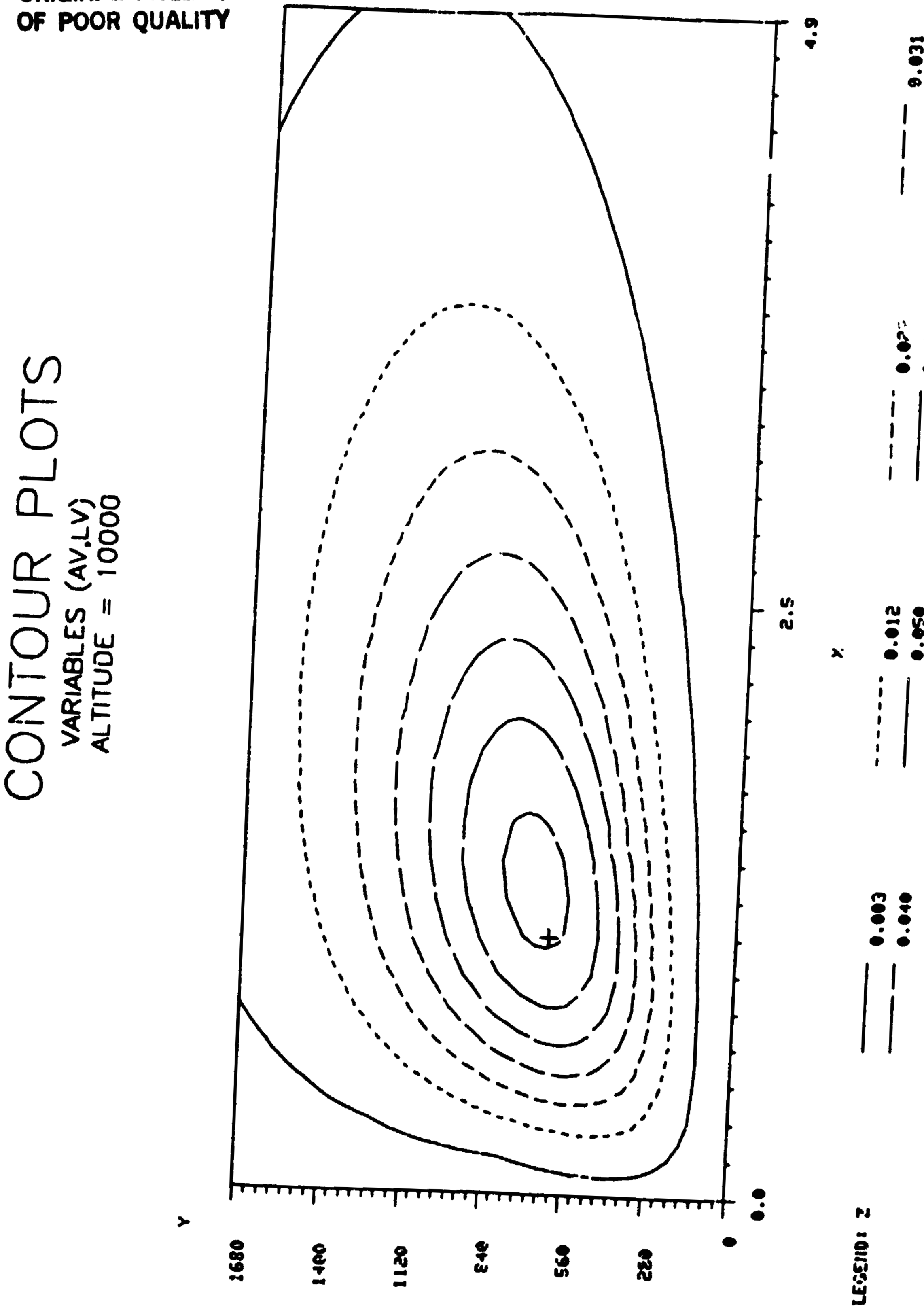

BIVARIATE GAMMA DENSITY

VARIABLES (AU,I U)
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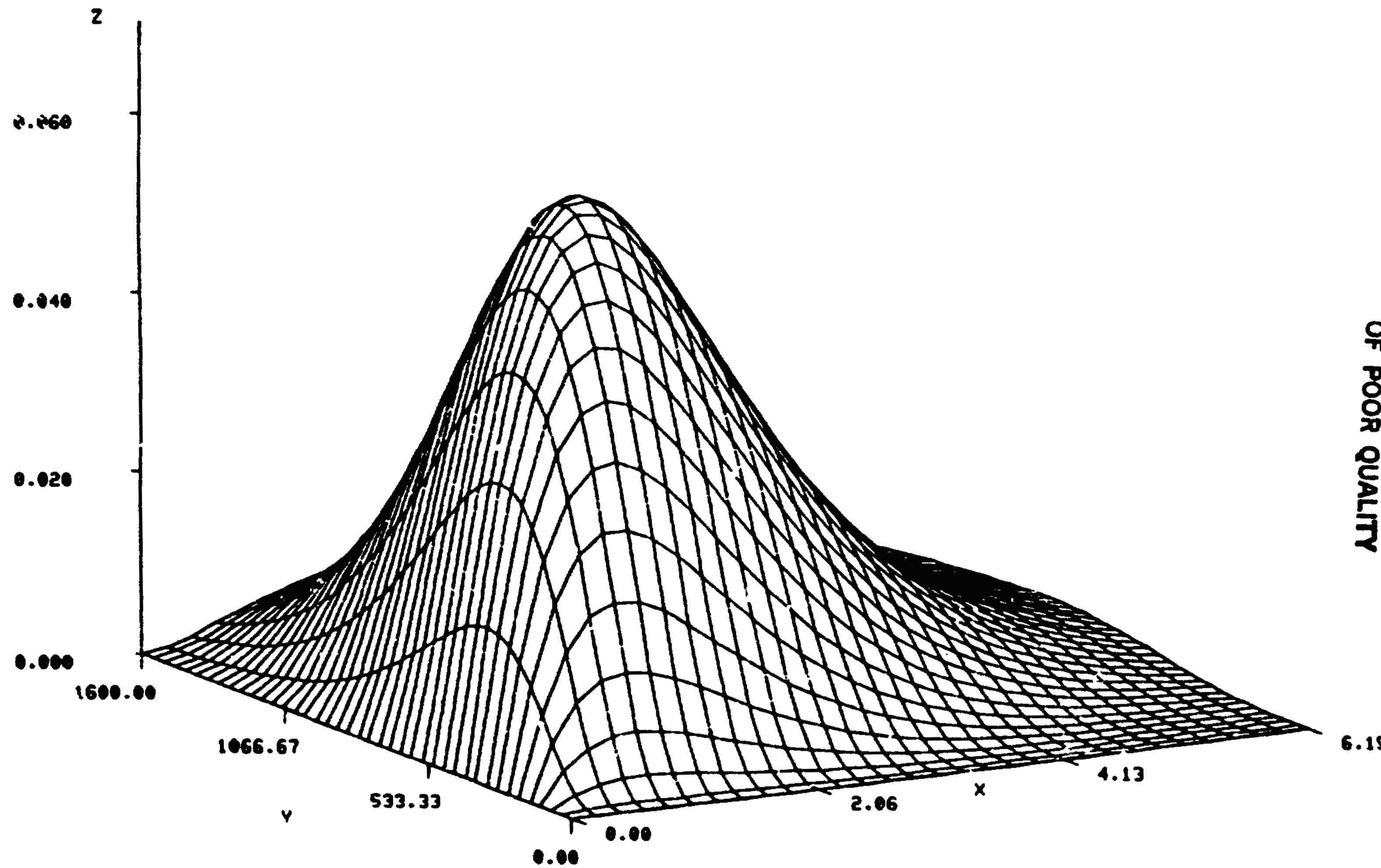
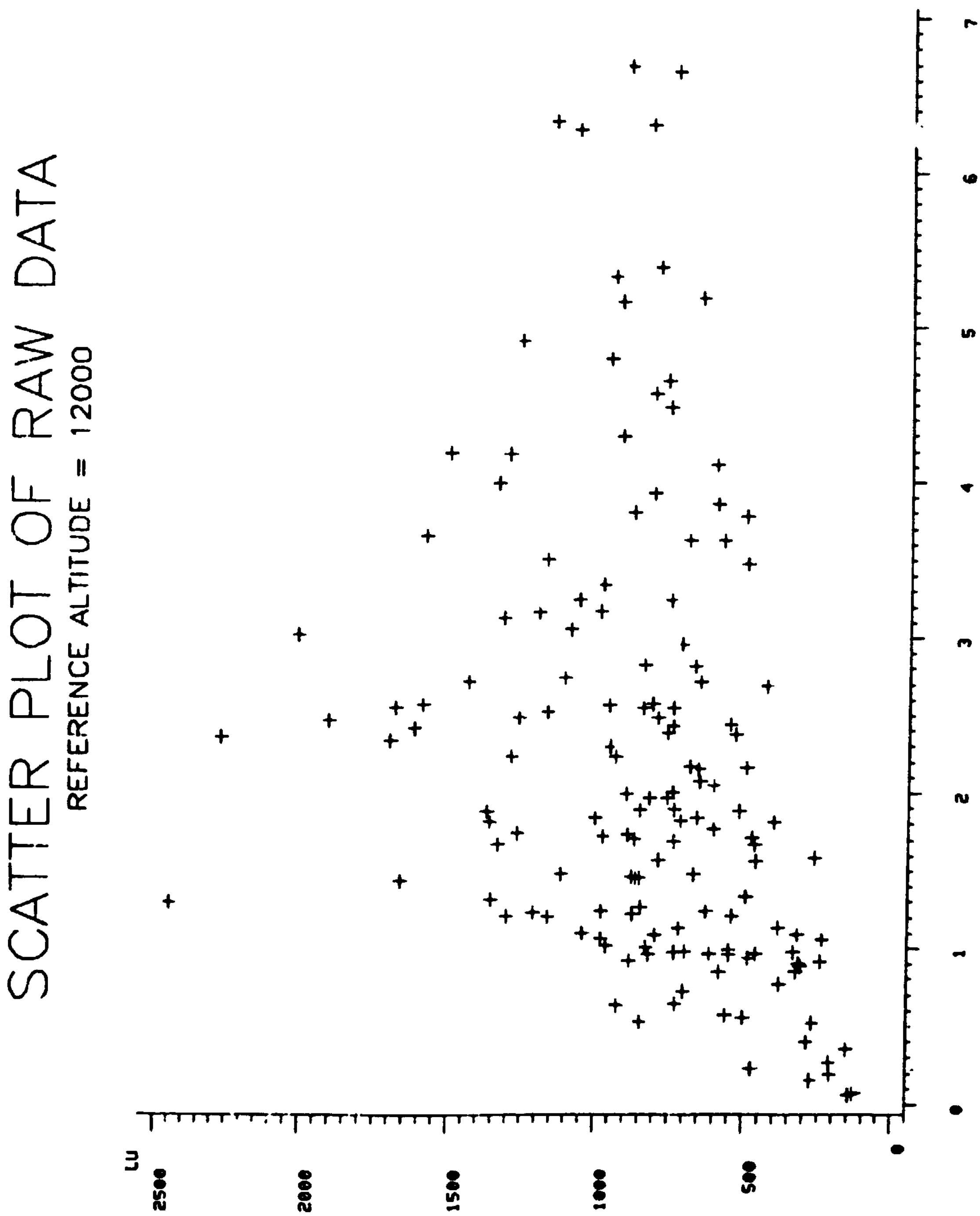
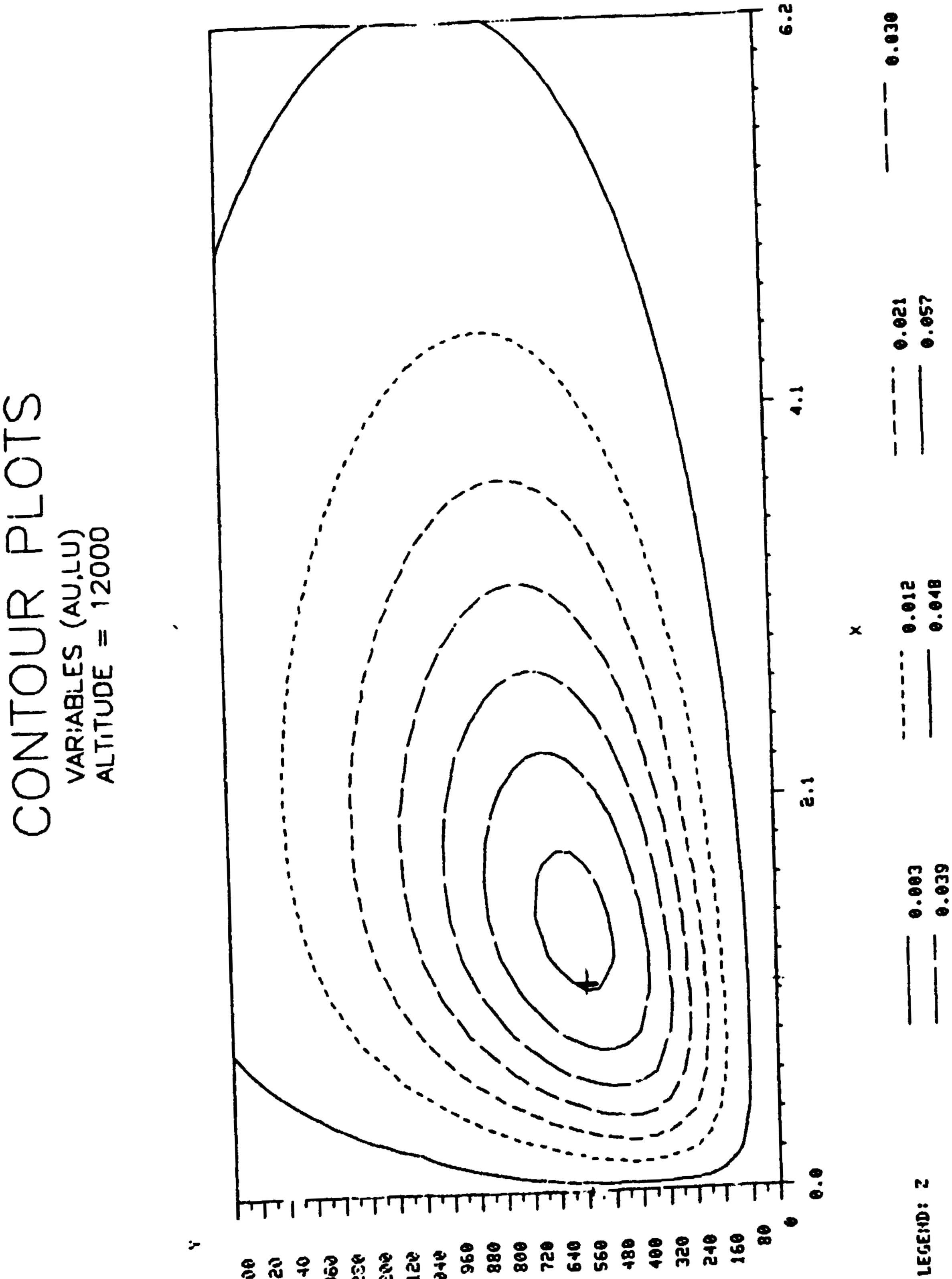



SCATTER PLOT OF RAW DATA
REFERENCE ALTITUDE = 12000
LU
2500
2000
1500
1000
500
0
1
2
3
4
5
6
7
AU


CONTOUR PLOTS
VARIABLES (AU,LU)
ALTITUDE = 12000
X
Y
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2.1
4.1
6.2
0
80
160
240
320
400
480
560
640
720
800
880
960
1040
1120
1200
1280
1360
1440
1520
1600
LEGEND: 2
0.003
0.039
0.012
0.048
0.021
0.057
0.030

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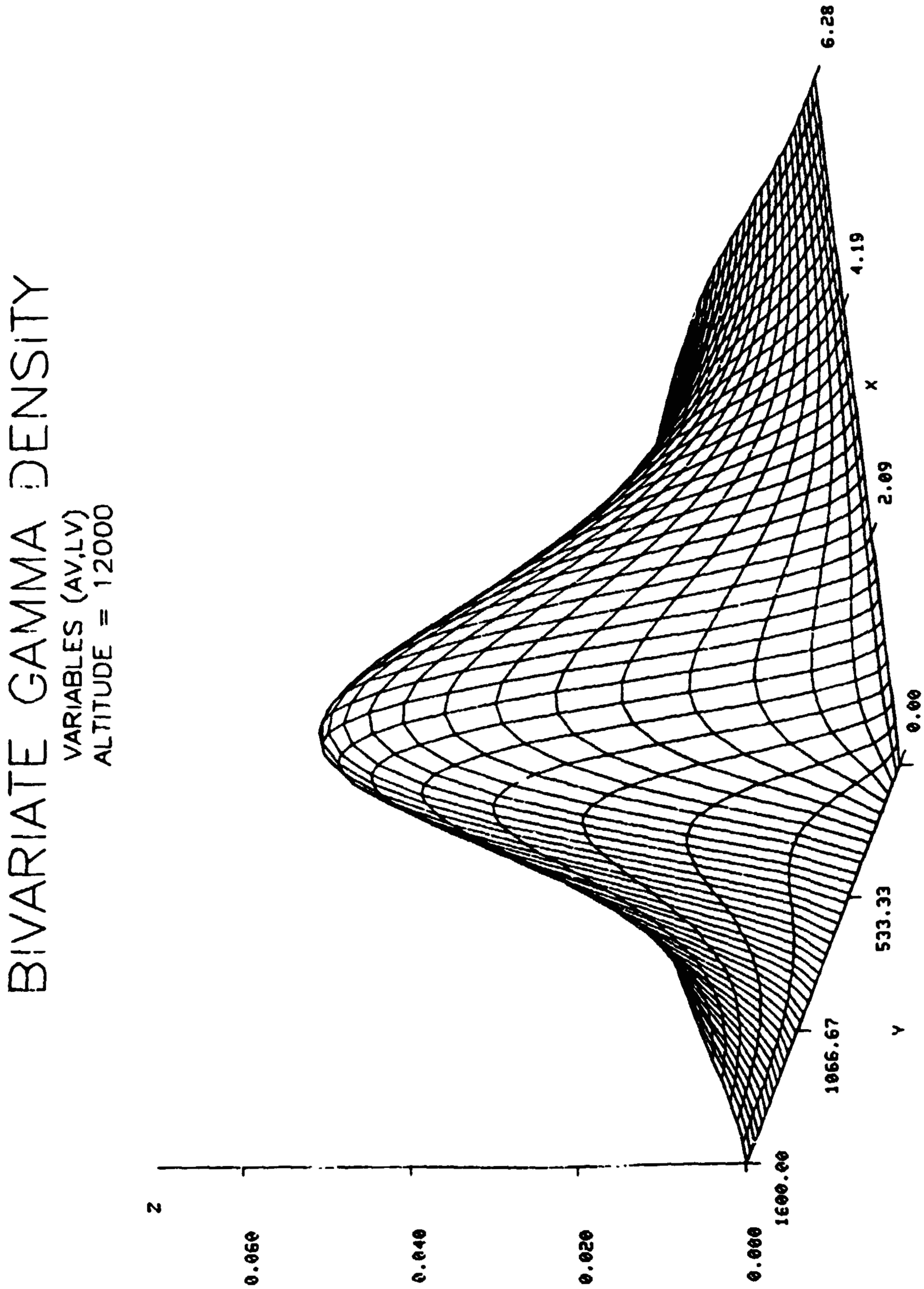

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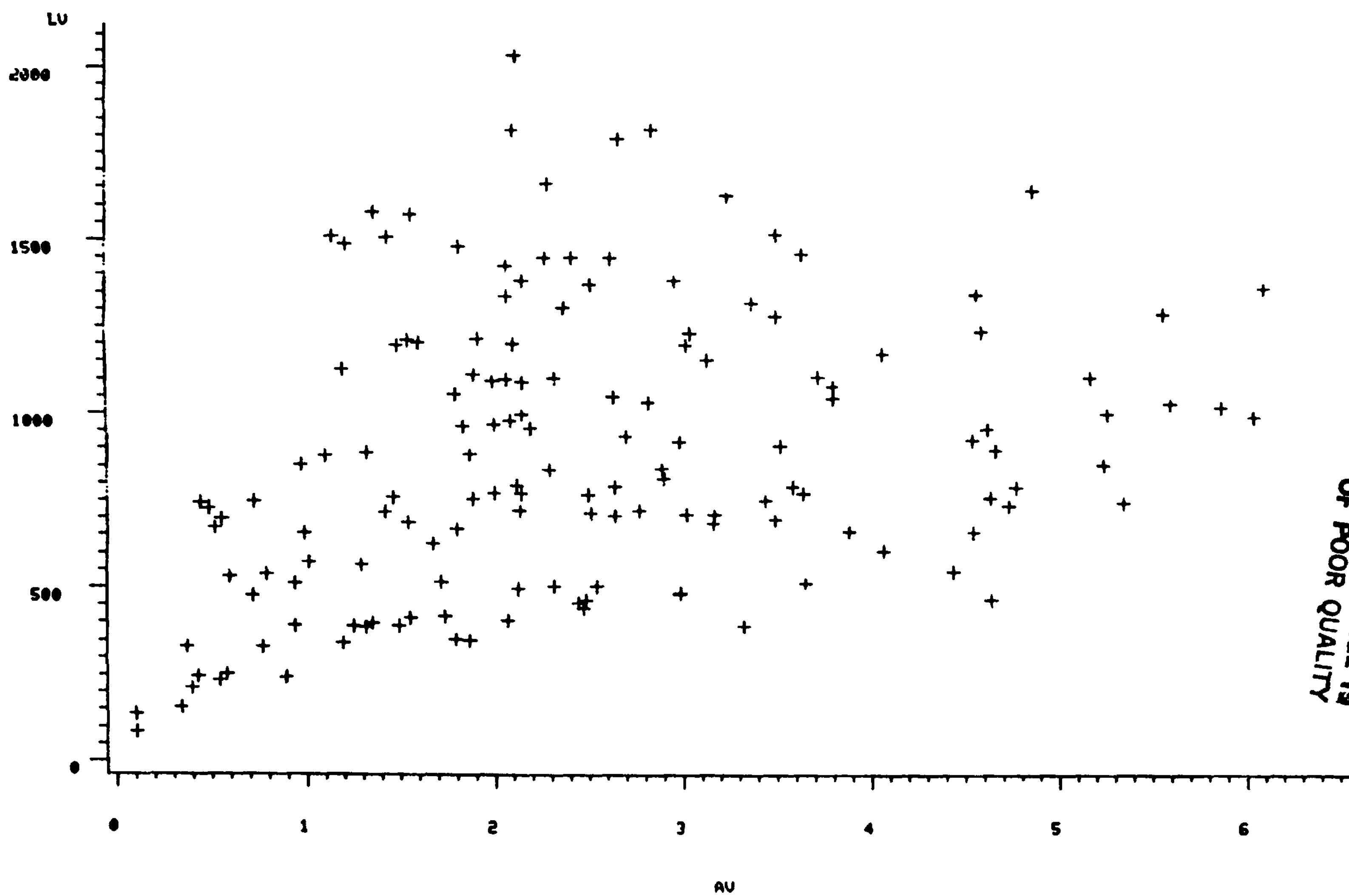

CONTOUR PLOTS

VARIABLES (AV,LV)
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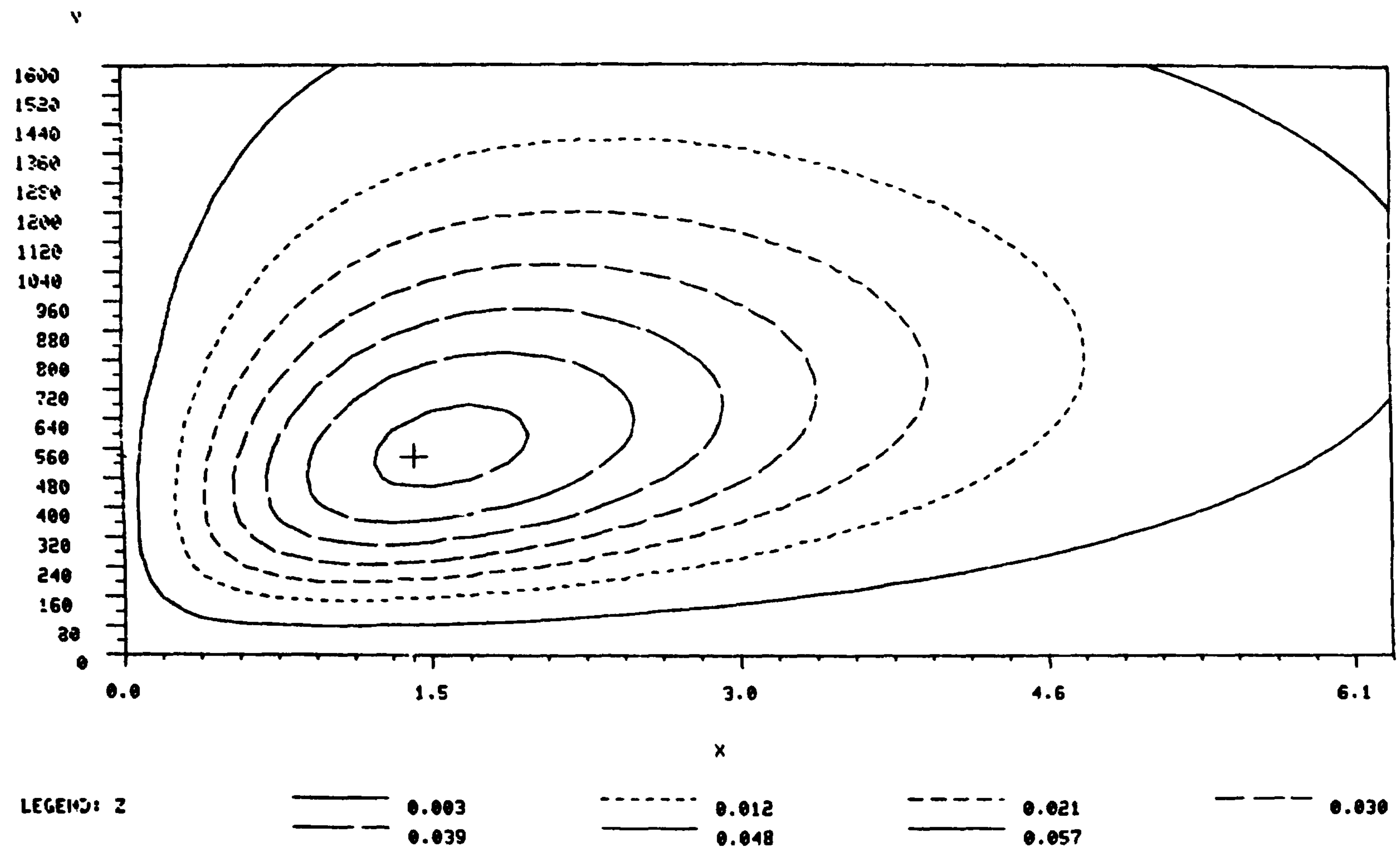

BIVARIATE GAMMA DENSITY
VARIABLES (AU,LU)
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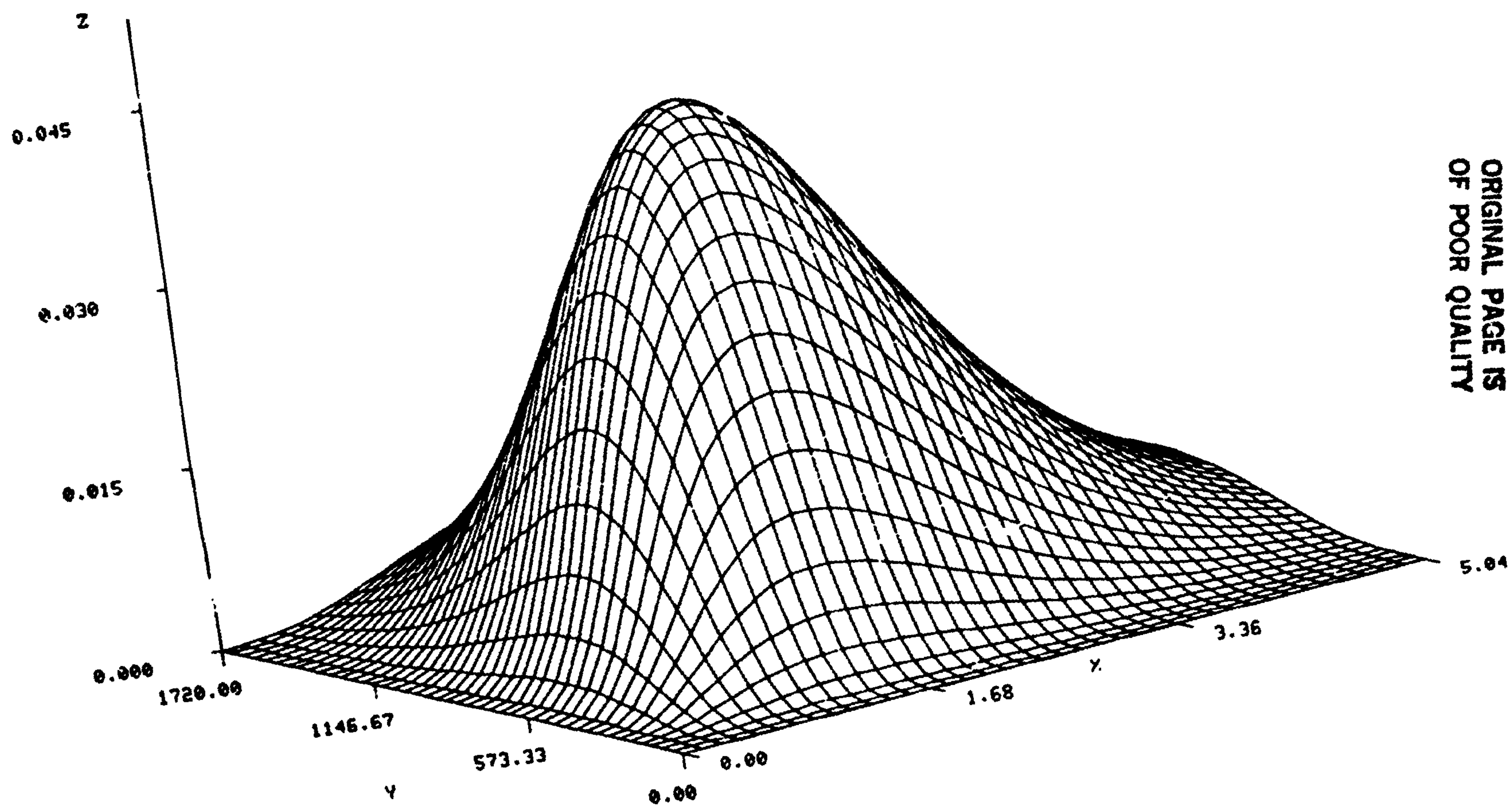

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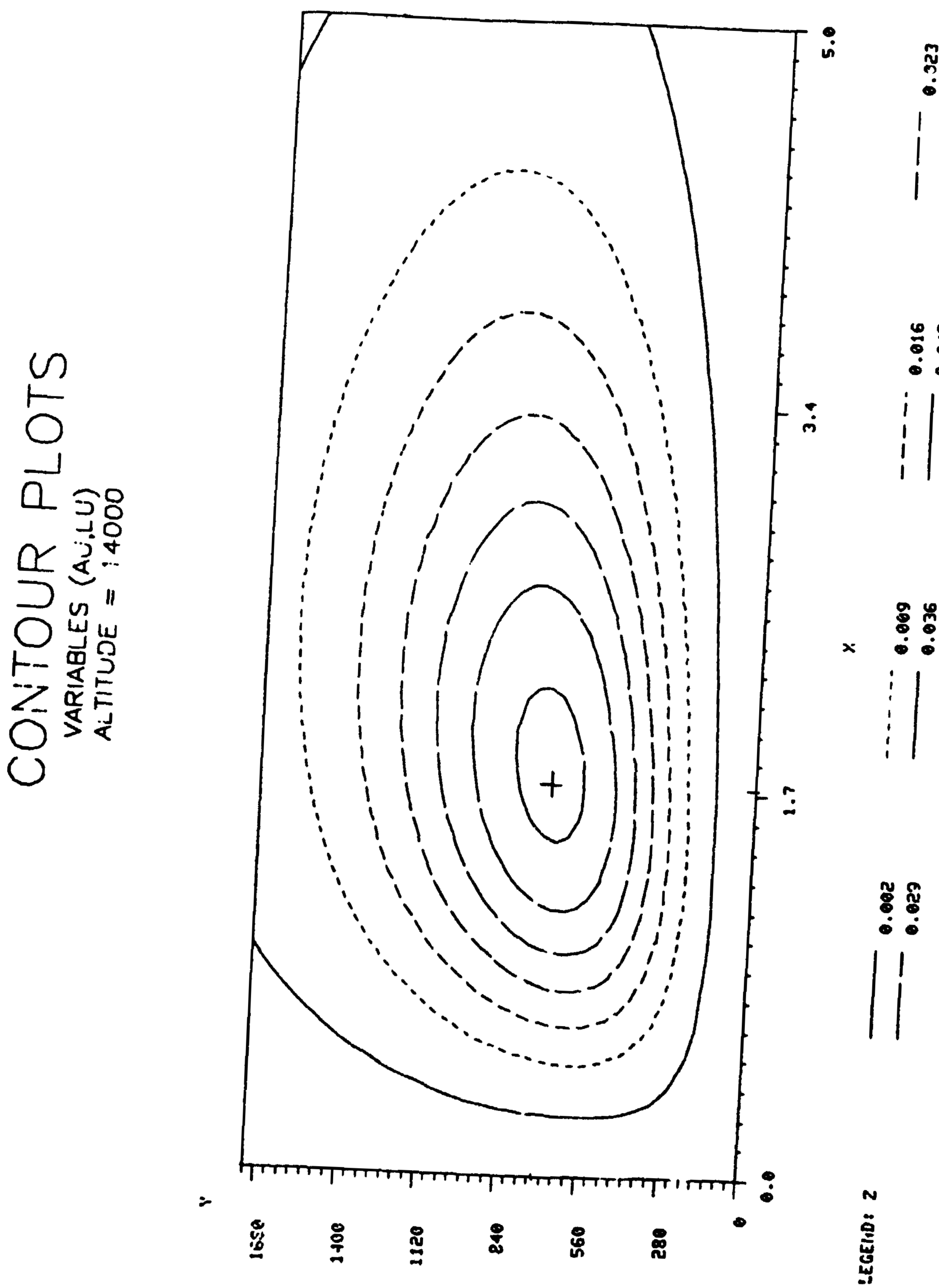

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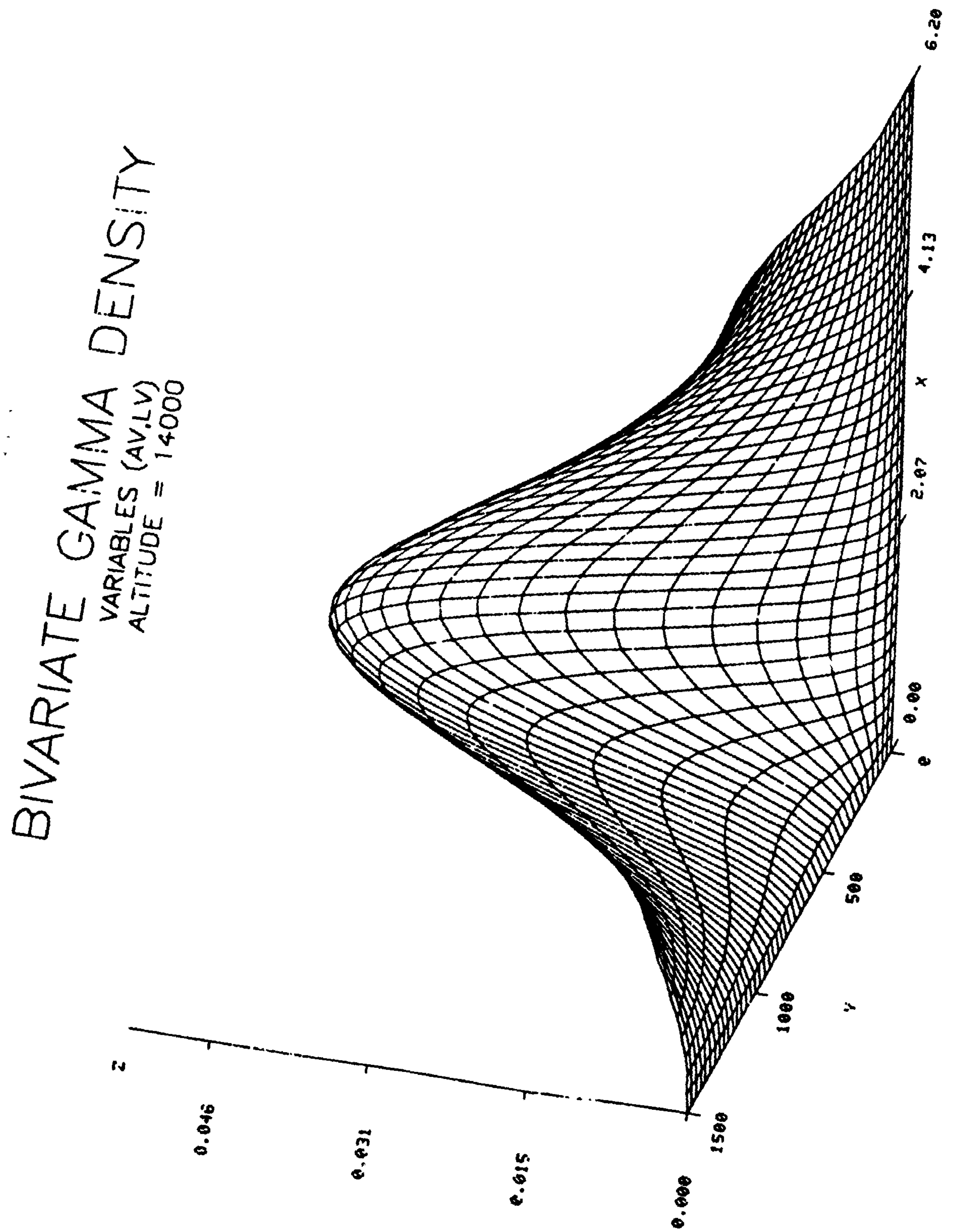

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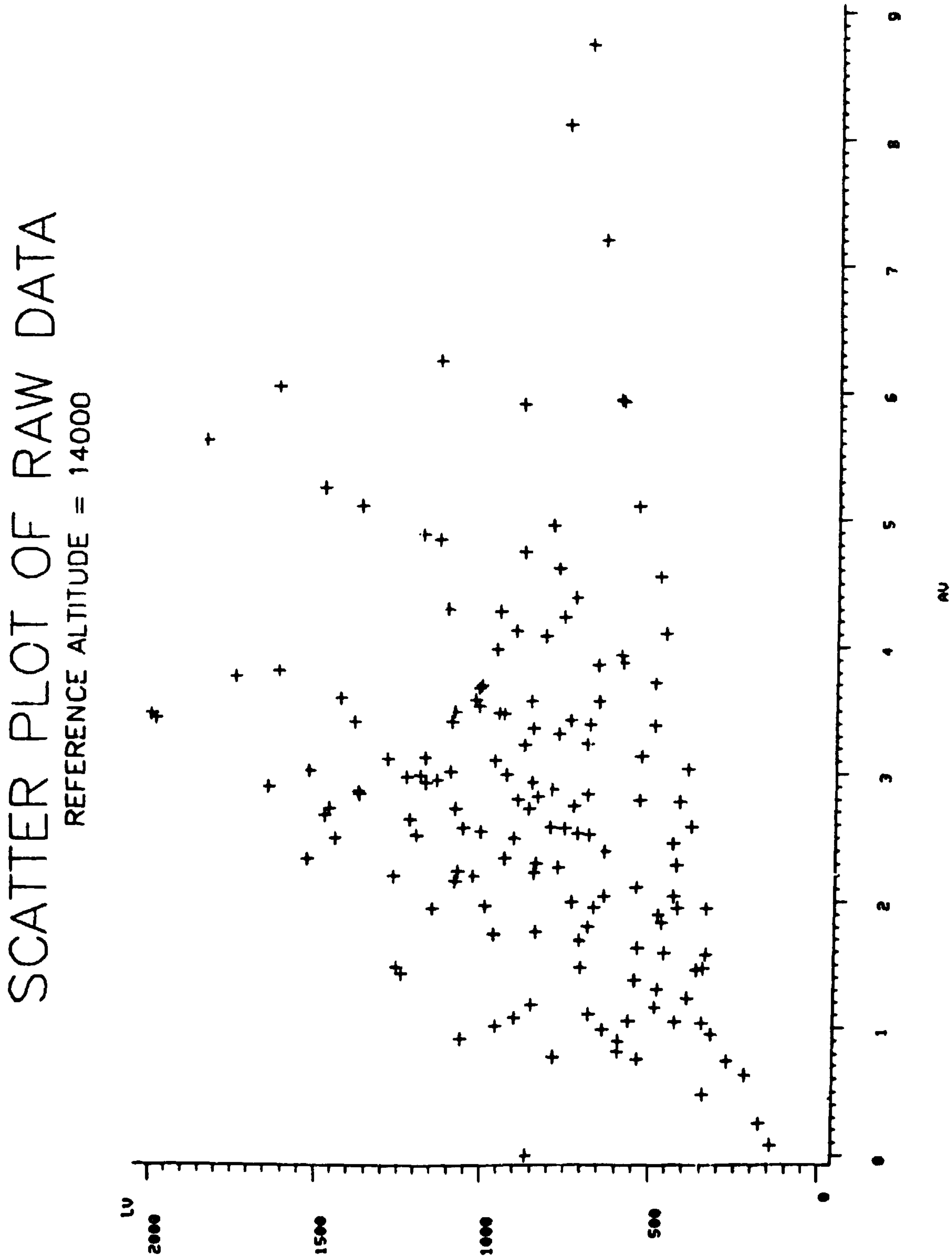

CONTOUR PLOTS

VARIABLES (AV,LV)
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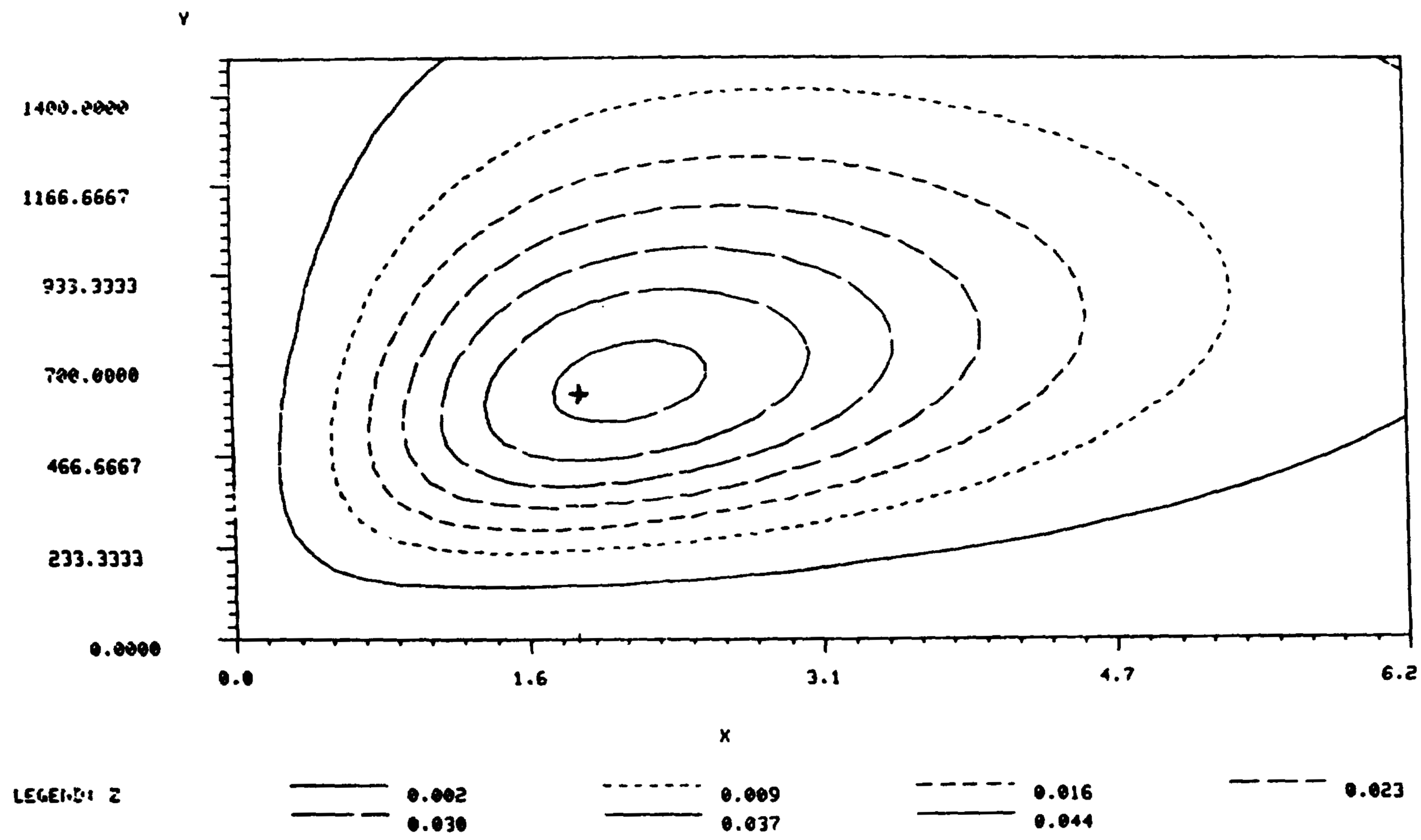

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